# A Disproof of the Riemann Hypothesis

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*Abstract* : *fhe Riemann Hypothesis (RH), concerning the non−trivial zeros of the Riemann zeta function, remains one of the most important unsolved problems in number theory.*

*fhis paper is the third and final one in a series of tmo previously published articles. fhese morks provide a rigorous demonstration that the Riemann hypothesis is false.*

*We approach this problem using a proof by contradiction. By assuming the Riemann Hypothesis to be true, me utilize the established principles of analytic and holomorphic functions and our previously stated initial conjecture to study the properties of the associated functions, namely ' and the expression* $S(\frac{1}{2}+ic)$, *such* $S(s) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s} = -\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = -\eta(s)$, $S(s) = {'}(s) S(1-s)$ *mith* ${'}(s) = 2\frac{1-2^{s-1}}{1-2^s}\pi^{s-1}\sin\frac{s\pi}{2}\Gamma(1-s) = g(s)e^{i8(s)}$ , $6s \in$ s *such* $\mathrm{Re}(s) \in ]0,1[$. *Our analysis demonstrates that if the Riemann Hypothesis mere true, it mould necessarily imply that the function ' is real and, furthermore, that the associated quantity* $S(1/2+ic)$ *must also be real. fhis conjunction of derived properties, homever, leads to a fundamental mathematical contradiction mithin the theory of the zeta function. fherefore, the initial hypothesis must be false. We thus rigorously demonstrate that the Riemann Hypothesis is false.*

*fhis refutation challenges a century−old conjecture and necessitates a re−evaluation of current theoretical framemorks concerning the distribution of prime numbers*

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*Keywords* : The Riemann Hypothesis, Dirichlet series, Zeta function, The Riemann conjecture, Adherent Point, Analytic functions.

2010 *MSC* : *11E45 (primary); 11M41 (secondary)*

# 1 Introduction

The Riemann Hypothesis is a conjecture formulated in 1859 by the mathematician Bernhard Riemann, according to which the nontrivial zeros of the Riemann zeta function are infinite and all have a real part equal to 1/2. See [6] [3]

His proof would improve knowledge of the distribution of prime numbers and open up new areas of mathematics. Riemann's article (see [4]) on the distribution of prime numbers is his only text dealing with number theory. He develops the properties of the zeta function $(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ and proves the prime number theorem by admitting several results, including what is now called the Riemann Hypothesis. Subsequently, Hardy demonstrated that there are infinitely many zeros on the critical line ($\mathrm{Re}\,(s) = \frac{1}{2}$). (Hardy's Theorem: see [5] [8], [9]), This gave us hope that the Riemann hypothesis might be true...

Given that the Riemann Hypothesis is currently unproven (it is one of the most famous unsolved problems in mathematics), stating that it is "false" implies that we have a refutation (a counterexample or a proof that it leads to a contradiction). In academic mathematics, it is essential to be very precise when questioning established conjectures.

That is why we delayed announcing our results, as we were looking for a formal and nuanced way to state that the Riemann Hypothesis (RH) is false in an English-speaking academic context.

In the two first paper [1] and [2] we have see: Preliminary: Analytic extension of the function (Riemann zeta function). Contributions: The first announcement (some lemma and the first claim). The second announcement (the second claim, and two conjectures). The first conjecture and The second conjecture. We wish to present them here before the proof and our new contribution.

This paper is the third and final one in a series of two previously published articles "A Contemporary Conjecture for the Riemann Hypothesis" (see [1]) and "New Proofs of the Equivalent Statement of the Dirichlet Eta Function and of the Riemann Hypothesis" (see [2]). This work provides a rigorous demonstration that the Riemann Hypothesis does not hold.

Let $S(s) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s} = -\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = -\eta(s)$, so $S(s) = g(s)\, e^{i8(s)} S(1-s)$.

Remark 1 *(Functional equation of Hardy):* $S(s) = {}'(s)\, S(1-s)$ $6s \in$ s *such* $\mathrm{Re}\,(s) \in\, ]0, 1[$

*mith* $'(s) = 2\frac{1-2^{s-1}}{1-2^s}\pi^{s-1} \sin\frac{s}{2}\pi\ \Gamma(1-s) = g(s)\, e^{i8(s)}$. *See [8] and [9].*

# 2 Preliminary

## 2.1 Analytic extension of the function

Theorem 2 *fhere exists a unique function, denoted (Riemann zeta function), verifying:*

- is meromorphic on the entire S, holomorphic outside a simple pole at $s = 1$, *mith residue* 1*;*
- $(s) = G(1, s)$, *if* $\mathrm{Re}(s) > 1$. *See [11] page 412.*

Proposition 3 *Get* $s = r + ic$, *so* $S = \sum_{n=1}^{+\infty} (-1)^n \frac{e^{-i\ln(n)c}}{n^r} = \sum_{n=1}^{+\infty} (-1)^n \frac{e^{i\ n}}{n^r}$ *is convergent for strictly positive real s, by application of the alternating series criterion; it is in fact the same for* $\mathrm{Re}(s) > 0$, *mhich is demonstrated using Abel's lemma (me can also shom more simply the absolute convergence of the serie* $\sum_{n=1}^{+\infty} \frac{(2n)^s-(2n-1)^s}{(2n)^s(2n-1)^s}$ *)*

*And fhe Riemann function is a meromorphic complex analytic function defined, for any complex number s such that* $\mathrm{Re}(s) > 1$, *by the Riemann serie:*

$$= \sum_{n=1}^{+\infty} \frac{1}{n^s} = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(n)c}}{(2n)^r} = \sum_{n=1}^{+\infty} \frac{e^{i\ n}}{(2n)^r} \Longrightarrow \ = \frac{S(s)}{2^{1-s}-1} = \frac{y(s)}{1-2^{1-s}}, \ mith$$

$_n = -\ln(n)c$.

Remark 4 *According to the theory of Dirichlet series, me deduce that the function thus defined is analytic over its domain of convergence. fhe series does not converge at* $s = 1$ *because me have* $\sum_{n=1}^{m} \frac{1}{n} \geq \int_1^{m+1} \frac{dx}{x} = \ln(m + 1)$.

*If* $\mathrm{Re}(s) > 1$, $(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = 2\sum_{n=1}^{+\infty} \frac{1}{(2n)^s} - S(s) = -S(s) + \frac{2}{2^s}\ (s)$,

*so* $(s) = \frac{S(s)}{2^{1-s}-1}$.

*fhis thus realizes the extension of the function over* $\mathrm{Re}(s) > 0$, *except for* $s = 1 + \frac{2k\pi}{\ln(2)}i$, $k \in \mathbb{Z}$.

Proposition 5 *Get* $s = r+ic = r+i\frac{}{\ln(2)}$ *(* $= \ln(2)c$*), so* $S = 2^{1-r}e^{-i} - 1$ . *(See [1] and [2])*

## 2.2 Introduction

All the results we will cited here are taken and slightly distorted from others already cited and demonstrated for a long time by other researchers and books. To demonstrate Lemma 1 we used the classic course of Complex Analysis like [12] and [7].

Definition 6 *(differentiability and holomorphy): A function* $f : U \to C$ *on an open set of the complex plane is said to be complex differentiable at the point* $z_0 \in U$ *if the limit* $f'(z_0) = lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ *exists. fhe function f is said to be holomorphic on U if it is complex differentiable at every point of U, and it is said to be holomorphic on a set A if there exists an open set V containing A on mhich f is defined and holomorphic. A holomorphic function on the entire* S *is called an entire function.*

Definition 7 *(analytic functions): We say that a function $f(z)$ defined on an open set $U$ of the complex plane* s *is analytic on $U$ if, at every point $z_0$, there exists a $6(z_0) > 0$ and complex numbers $c_n(z_0)$ such that for $|z - z_0| < 6(z_0)$, me have* $f(z) = \sum_{n=0}^{\infty} c_n(z_0)(z - z_0)^n$.

Theorem 8 *Every analytic function is holomorphic, and indeed infinitely dif− ferentiable in the complex sense; moreover, all its derived functions are also analytic functions.*

Remark 9 *Gauchy's theory leads us to a fundamental theorem: every holomor− phic function is an analytic function. In particular, every holomorphic function is automatically infinitely differentiable in the complex sense.*

Theorem 10 *1− If $f$ and $g$ are holomorphic on $U$ and coincide on a set mith a non−isolated point, they are equal.*

*2− If the set of points of a domain $U$ mhere tmo analytic functions $f$ and $g$ on $U$ take the same values has an accumulation point in $U$, then in fact the tmo functions $f$ and $g$ take the same values everymhere in $U$. fhey are, in fact, the same functions.*

Theorem 11 *(Adherent Point and Glosure): Get $X$ be a topological space and $A \subseteq X$ be a subset. A point $x \in X$ is said to be an adherent point (Glosure point) of $A$ if every open neighborhood of $x$ intersects $A$. fhe closure of $A$, denoted by $\overline{A}$, consists of all adherent points of $A$.*

Theorem 12 *(Adherent Point and Existence of Gonvergent Sequences): Get $X$ be a topological space and $A \subseteq X$ be a subset. A point $x \in X$ is an adherent point (Glosure point) of $A$ if and only if there exists a sequence $x_n$ in $A$ such that:* $\lim x_n = x$ .

# 3 Our previous contributions

This section contains antecedent results cited and demonstrated in the two previously published articles "A contemporary conjecture for the Riemann hypothesis" (see [1]) and "New proofs of the equivalent statement of the Dirichlet eta function and the Riemann hypothesis" (see [2]).

## 3.1 The first announcement

Lemma 13 *Assuming that there exists an $s_1$ mith $r_1 = \mathrm{Re}[s_1] \in \ 0, \frac{1}{2}$ and* $c = \mathrm{Im}[s_1] > 0$ *such that $S^2(s_1) \in IR$, so*

(i) $\mathsf{I} V(s_1) \subset$ s *such* $6s \in V(s_1) - \{s_1\}$, $S^2(s) \notin IR$ __________

(ii) $\mathsf{I} u_n \in V(s_1) - \{s_1\}$ *such* $\lim u_n = s_1$ *(since $s_1 \in V(s_1) - \{s_1\}$ mith $\overline{A}$ is the adherant of A).*

Claim 14 *Get* $D = \{z \in$ s$/ \mathrm{Re}(z) \in ]0, 1[\}$, *so* $6s \in D$,

$S(s) \in IR$ **e** $S(1 - s) \in IR$ *and,*

$S(s) \in iIR$ **e** $S(1 - s) \in iIR$.

## 3.2 The second announcement

Claim 15 $\exists s_0 / S(s_0) \in iIR \Longleftrightarrow \exists s_1 \in (r_0, s_0] / S(s_1) = 0.$

*Such* $r_0 = \mathrm{Re}(s_0) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[$ *and* $(r_0, s_0] = \{r_0 + ic \in s / 0 < c \leq c_0\}$

Corollary 16 $\exists m_0, \exists s_0$ *such* $S^{m_0}(s_0) \in iIR \Longleftrightarrow \exists s_1 \in (r_0, s_0]$ *such* $S(s_1) = 0$
$\Longleftrightarrow \exists c \in ]0, c_0]$ *such* $S(r_0 + ic) = 0$
*mith* $r_0 = \mathrm{Re}(s_0) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[$ *and* $(r_0, s_0] = \{r_0 + ic \in s / 0 < c \leq c_0\}$

Conjecture 17 $\forall r \in ]0, 1[$ *and* $s = r + ic$

*1)* $r \neq \frac{1}{2} \Rightarrow \mathrm{Re}[S(s)] \neq 0$

*2)* $\mathrm{Re}[S(s)] = 0 \Longrightarrow r = \frac{1}{2}$

*3)* $\sum_{n=1}^{+\infty} (-1)^n \frac{\cos[\ln(n)c]}{n^r} = 0 \Longrightarrow r = \frac{1}{2}$

Conjecture 18 *(* $r = \frac{1}{2}$ *and* $\theta\left(\frac{1}{2} + ic\right) = (2k+1)\pi$ *)* $\Longrightarrow S\left(\frac{1}{2} + ic\right) = 0$

# 4 News contributions "The Riemann conjecture is not true!!"

## 4.1 Introduction

We will proceed (with the reasoning) by contradiction: Assuming that the Riemann conjecture is true.

So $S(s) \neq 0 \; \forall s \in D$, with $D = \left\{s \in s \,/\, r = \mathrm{Re}[s] \in \left]0, \frac{1}{2}\right[ \text{ and } c = \mathrm{Im}[s] > 0\right\}$.
According to our initial conjecture that we had announced earlier:
$r \neq \frac{1}{2} \Rightarrow \mathrm{Re}[S(s)] \neq 0$
$\Longrightarrow \mathrm{Re}[S(s)] \neq 0 \quad \forall s \in D.$
Let $s_1 \in D$, so $\mathrm{Re}[S(s_1)] \neq 0$

$\Longrightarrow \exists \alpha \in IR$ such $S(s_1) e^{i\alpha} \in iIR_*$.
Let $A(s) = S(s) e^{i\alpha}$ and $B(s) = S(1-s) e^{i\alpha}$
so, $A(s_1) = S(s_1) e^{i\alpha} \in iIR_*$ and $A(s) = \chi(s) B(s)$, as $S(s) = \chi(s) S(1-s)$.

Lemma 19 *Assuming that there exists an* $s_1$ *mith* $r_1 = \mathrm{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ *and* $c = \mathrm{Im}[s_1] > 0$ *such that* $A^2(s_1) \in IR$, *so*

(i) $\exists V(s_1) \subset s$ *such* $\forall s \in V(s_1) - \{s_1\}$, $A^2(s) \notin IR$

(ii) $\exists u_n \in \underline{V(s_1) - \{s_1\}}$ *such* $\lim u_n = s_1$

*(since* $s_1 \in \overline{V(s_1) - \{s_1\}}$ *mith* $\overline{V}$ *is the adherant of* $V$*).*

Proof. $(i)$ Reasoning by the absurd. Suppose: $\forall V(s_1) \subset s$, $\exists s \in V(s_1) - \{s_1\}$ such $A^2(s) \in IR$ (*).

Let $s(0) \neq s_1$ such $s(0)$ very close to $s_1$, and $V_0(s_1)$ such $s(0) \in V_0(s_1)$
$\Longrightarrow \exists s(1) \in V_0(s_1) - \{s_1\}$ such $A^2[s(1)] \in IR$
$s(1) \neq s_1 \Longrightarrow \exists V_1(s_1)$ such $s(1) \notin V_1(s_1)$ and $V_1(s_1) \subset V_0(s_1)$

$\Longrightarrow \exists s(2) \in V_1(s_1) - \{s_1\}$ such $A^2[s(2)] \in IR$

...

and so on

if: $\exists V_{n-1}(s_1)$ such $s(n-1) \notin V_{n-1}(s_1)$ and $V_{n-1}(s_1) \quad V_{n-2}(s_1)$

and $\exists s(n) \in V_{n-1}(s_1) - \{s_1\}$ such $A^2[s(n)] \in IR$

$\downarrow$

$\exists V_n(s_1)$ such $s(n) \notin V_n(s_1)$ and $V_n(s_1) \quad V_{n-1}(s_1)$

$\Longrightarrow \exists s(n+1) \in V_n(s_1) - \{s_1\}$ such $A^2[s(n+1)] \in IR$

so, we construct a sequence $s(n)$ which converges to $s_1$ such that $A^2(s(n)) \in IR$,

$\lim V_n(s_1) = \lim \bigcap_{h=0}^{n} V_h(s_1) = \{s_1\}$, since $\{s_1\} \quad V_n(s_1) \quad V_{n-1}(s_1)\ \forall n$.

Consequently: $A^2(s(n)) - \overline{A^2}(s(n)) = 0$

$A^2$ and $\overline{A^2}$ are analytic and holomorphic functions, because, if $f$ satisfies the Cauchy-Riemann equations then $\overline{f}$ also satisfies it (Using the Cauchy-Riemann equations and Schwarz's theorem).

So $A^2(s) - \overline{A^2}(s)$ is analytic and holomorphic function.

Let $U = \{s(n)/n \in IN\} \cup \{s_1\}$, so $A^2(s) - \overline{A^2}(s)$ and $0$ are two analytic functions take the same values on $U$, and $U$ has an accumulation point (in $U$). Hence, $A^2(s) - \overline{A^2}(s) = 0$

$\Longrightarrow A^2(s) \in IR \longrightarrow$ Absurd!!

(*ii*) Using (*i*) and the theorem (Adherent Point and Existence of Convergent Sequences): Let $X$ be a topological space and $A \subseteq X$ be a subset. A point $x \in X$ is an adherent point (Closure point) of $A$ if and only if there exists a sequence $x_n$ in $A$ such that $\lim x_n = x$. ■

## 4.2 The absurdity:

Lemma 20 *The Riemann conjecture is true* $\Longrightarrow$ $'(s) \in IR$, $\forall s \in D$.

*With* $D = \left\{ s \in s/r = \text{Re}[s] \in \left]0, \frac{1}{2}\right[ \text{ and } c = \text{Im}[s] > 0 \right\}$

Proof. Let $s_1 \in D$, we have seen if The Riemann conjecture is true, so $\text{Re}[S(s_1)] \neq 0$

$\Longrightarrow \exists \ \in IR$ such $S(s_1)e^{i} \in iIR_*$.

Let $A(s) = S(s)e^{i}$ and $B(s) = S(1-s)e^{i}$

so $A(s_1) = S(s_1)e^{i} \in iIR_*$, $A^2(s_1) \in IR_*^-$

and $A(s) = '(s)B(s) = g(s)e^{i\delta(s)}B(s)$

since $S(s) = '(s)S(1-s)$ and $'(s) = g(s)e^{i\delta(s)}$

Since the last lemma:

Assuming that $\exists s_1$ with $r_1 = \text{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ and $c > 0 / A^2(s_1) \in IR$, so

(i) $\exists V(s_1) \subset s$ such $\forall s \in V(s_1) - \{s_1\}$, $A^2(s) \notin IR$

(ii) $\exists u_n \in V(s_1) - \{s_1\}$ such $\lim u_n = s_1$.

$\lim u_n = s_1 \Longrightarrow \exists N \in IN$ such $\forall n \geq N$, $u_n \in V(s_1) - \{s_1\}$

$\Rightarrow \exists N \in IN$ such $\forall n \geq N$, $A^2(u_n) \notin IR \Rightarrow \left(A(u_n), \overline{A(u_n)}\right)$ is a basis of s

$\Rightarrow \exists! (a_n, b_n) \in IR^2$ such $B(u_n) = a_n A(u_n) + b_n \overline{A(u_n)}$

$A(s) = '(s)B(s) \Rightarrow B(u_n) = a_n '(u_n)B(u_n) + b_n '(u_n)B(u_n)$

$$\Rightarrow \left[1-a_n\varphi'(u_n)\right] B(u_n) = \left[b_n\varphi'(u_n)\right] \overline{B(u_n)}$$

$$\Rightarrow \left[\overline{1-a_n\varphi'(u_n)}\right] \overline{B(u_n)} = [b_n\varphi'(u_n)] B(u_n)$$

$$\Rightarrow b_n\varphi'(u_n) B^2(u_n) = \left[1-\overline{a_n\varphi'(u_n)}\right] |B(u_n)|^2$$

$\varphi'(s)\varphi'(1-s) = 1, \forall s \Rightarrow b_n B^2(u_n) = \varphi'(1-u_n)\left[1-\overline{a_n\varphi'(u_n)}\right] |B(u_n)|^2$

$$\Rightarrow b_n B^2(u_n) = \left[\varphi'(1-u_n) - a_n\overline{\varphi'(u_n)}\varphi'(1-u_n)\right] |B(u_n)|^2$$

$\varphi'(s)\varphi'(1-s) = 1 \Rightarrow |\varphi'(s)|^2 \varphi'(1-s) = \overline{\varphi'(s)}$

and $\varphi'(s) = g(s)e^{i\theta(s)} \Rightarrow \overline{\varphi'(s)} = g(s)^2\varphi'(1-s)$

$\Rightarrow b_n B^2(u_n) = |B(u_n)|^2 \left[\varphi'(1-u_n) - a_n g_n^2\varphi'^2(1-u_n)\right]$

$\Rightarrow |b_n| = |\varphi'(1-u_n)|\left|1-a_n g_n^2\varphi'(1-u_n)\right|$ or $|B(u_n)| = 0$

we have $A^2(u_n) \notin \mathbb{R} \Rightarrow A(u_n) \neq 0 \Rightarrow |B(u_n)| \neq 0$

since $A(s) = g(s)e^{i\theta(s)}B(s)$

$\Rightarrow |b_n| = |\varphi'(1-u_n)|\left|1-a_n g_n^2\varphi'(1-u_n)\right|$

also $g_n \neq 0$ since $|A(u_n)| = g_n|B(u_n)|$

$\Rightarrow |b_n| = \frac{1}{\rho_n}\left|1-a_n g_n^2\varphi'(1-u_n)\right| \Rightarrow b_n^2 g_n^2 = \left|1-a_n g_n^2\varphi'(1-u_n)\right|^2$

$\Rightarrow$

$$b_n^2 g_n^2 = 1 + a_n^2 g_n^2 - 2a_n g_n\cos(\theta_n) \qquad (1)$$

$B(u_n) = a_n A(u_n) + b_n\overline{A(u_n)} \Rightarrow$

$|B(u_n)|^2 = \left(a_n^2 + b_n^2\right)|A(u_n)|^2 + a_n b_n\left(A^2(u_n) + \overline{A^2(u_n)}\right) \Rightarrow$

$g_n^2|B(u_n)|^2 = \left(a_n^2 g_n^2 + b_n^2 g_n^2\right)|A(u_n)|^2 + a_n b_n g_n^2\left(A^2(u_n) + \overline{A^2(u_n)}\right) \Rightarrow$

$|A(u_n)|^2 = \left(a_n^2 g_n^2 + 1 + a_n^2 g_n^2 - 2a_n g_n\cos\theta_n\right)|A(u_n)|^2 + a_n b_n g_n^2\left(A^2(u_n) + \overline{A^2(u_n)}\right)$

$\Rightarrow 0 = \left[2a_n^2 g_n^2 - 2a_n g_n\cos\theta_n\right]|A(u_n)|^2 + a_n b_n g_n^2\left(A^2(u_n) + \overline{A^2(u_n)}\right)$

$\Rightarrow a_n g_n\left[2(a_n g_n - \cos\theta_n)|A(u_n)|^2 + b_n g_n\left(A^2(u_n) + \overline{A^2(u_n)}\right)\right] = 0$

$\Rightarrow a_n g_n = 0$ or $2[a_n g_n - \cos\theta_n]|A(u_n)|^2 + b_n g_n\left(A^2(u_n) + \overline{A^2(u_n)}\right) = 0$

we have $g_n \neq 0$ since $A(u_n) = g_n e^{i\theta_n}B(u_n) \neq 0$ $\left(A^2(u_n) \notin \mathbb{R}\right)$

$\Longrightarrow a_n = 0$ or $2[a_n g_n - \cos\theta_n]|A(u_n)|^2 + b_n g_n\left(A^2(u_n) + \overline{A^2(u_n)}\right) = 0$

1st case: If $\exists N_1 \in \mathbb{N}$ such $\forall n \geq N_1$ $a_n \neq 0$

so $\forall n \geq N_1$, $2[a_n g_n - \cos\theta_n]|A(u_n)|^2 + b_n g_n\left[A^2(u_n) + \overline{A^2(u_n)}\right] = 0$

$\Longrightarrow b_n g_n\left[A^2(u_n) + \overline{A^2(u_n)}\right] = -2[a_n g_n - \cos\theta_n]|A(u_n)|^2$

$\Rightarrow b_n^2 g_n^2\left[A^2(u_n) + \overline{A^2(u_n)}\right]^2 = 4[a_n g_n - \cos\theta_n]^2|A(u_n)|^4 \Rightarrow$

$\left[1 + a_n^2 g_n^2 - 2a_n g_n\cos\theta_n\right]\left[A^2(u_n) + \overline{A^2(u_n)}\right]^2 = 4[a_n g_n - \cos\theta_n]^2|A(u_n)|^4 \Rightarrow$

$\left[(a_n g_n - \cos\theta_n)^2 + \sin^2(\theta_n)\right]\left[A^2(u_n) + \overline{A^2(u_n)}\right]^2 = 4[a_n g_n - \cos\theta_n]^2|A(u_n)|^4 \Rightarrow$

$\sin^2(\theta_n)\left[A^2(u_n) + \overline{A^2(u_n)}\right]^2 = [a_n g_n - \cos\theta_n]^2\left[4|A(u_n)|^4 - \left(A^2(u_n) + \overline{A}^2(u_n)\right)^2\right]$

$\Rightarrow \sin^2(\theta_n)\left[A^2(u_n) + \overline{A^2(u_n)}\right]^2 = [a_n g_n - \cos\theta_n]^2\left[2|A(u_n)|^4 - A^4(u_n) - \overline{A}^4(u_n)\right]$

$\Rightarrow \sin^2(\theta_n)\left[A^2(u_n)+\overline{A^2(u_n)}\right]^2 = -[a_n g_n - \cos\theta_n]^2\left[A^2(u_n)-\overline{A^2(u_n)}\right]^2$

$\Rightarrow [i\sin\theta_n]^2\left[A^2(u_n)+\overline{A^2(u_n)}\right]^2 = [a_n g_n - \cos\theta_n]^2\left[A^2(u_n)-\overline{A^2(u_n)}\right]^2$

$\Rightarrow [a_n g_n - \cos\theta_n]\left[A^2(u_n)-\overline{A^2(u_n)}\right] = i\sin(\theta_n)\left[A^2(u_n)+\overline{A^2(u_n)}\right] \Rightarrow$

$[a_n g_n - \cos(\theta_n) \quad i\sin(\theta_n)]A^2(u_n) = [a_n g_n - \cos(\theta_n) \quad i\sin(\theta_n)]\overline{A^2(u_n)}$

so, $Z = \overline{Z} = [a_n g_n - \cos(\theta_n) \quad i\sin(\theta_n)]A^2(u_n) \in \mathbb{R}$

$\Rightarrow \left(a_n g_n - e^{i\theta_n}\right)A^2(u_n) \in \mathbb{R}$.

Let $A^2(u_n) = R_n + iI_n$, with $(R_n, I_n) \in IR^2$

$\Longrightarrow \left(a_n g_n - e^{i\theta_n}\right)A^2(u_n) = [a_n g_n - \cos(\theta_n) \quad i\sin(\theta_n)](R_n + iI_n) \in IR$

$\Longrightarrow R_n(a_n g_n - \cos\theta_n) \quad I_n\sin\theta_n + i[I_n(a_n g_n - \cos\theta_n) \quad R_n\sin(\theta_n)] \in IR$

$\Longrightarrow I_n(a_n g_n - \cos\theta_n) \quad R_n\sin(\theta_n) = 0$

we have $A(s) = S(s)e^{i}$

$\Longrightarrow A(u_n) = S(u_n)e^{i}$ and $s \mapsto A(s)$ is continuous

$\Longrightarrow \lim(R_n + iI_n) = \lim A^2(u_n) = \lim(R_n) + i\lim(I_n) = A^2(s_1) \in IR^-_*$

$\Longrightarrow \lim I_n = 0 \Longrightarrow \lim(R_n\sin\theta_n) = 0$

and $\lim(R_n) \neq 0$ (exists and is not zero)

$\Longrightarrow \lim(\sin\theta_n) = 0$

we have $\theta_n = \theta(u_n) \Longrightarrow \lim(\theta_n) = \lim\theta(u_n) = \theta(s_1)$

$\Longrightarrow \theta(s_1) = k\pi\ (k \in \mathbb{Z}) \Longrightarrow \varphi(s_1) = g(s_1)e^{i\theta(s_1)} = g(s_1)e^{ih\nu}$

$\Longrightarrow \varphi(s_1) \in IR$

2d case: If $\forall m \in IN$, $\exists n \geq m$ such $a_n = 0$.

Let $n = \$(m)$, so $\forall m \in IN$ $a_{\$(m)} = 0$

$m \longrightarrow a_{\$(m)} = 0$ is a sequence extracted of $(a_h)_{h \in IN}$

We have $B(u_n) = a_n A(u_n) + b_n\overline{A(u_n)}$ so $B\left(u_{\$(m)}\right) = b_{\$(m)}\overline{A\left(u_{\$(m)}\right)}$

$\left\{ A\left(u_{\$(m)}\right) = \varphi\left(u_{\$(m)}\right) b_{\$(m)}\overline{A\left(u_{\$(m)}\right)} \right.$

$\Longrightarrow \quad b_{\$(m)} = \frac{B(u_{\$(m)})}{A(u_{\$(m)})}$ since $A(s) = \varphi(s)B(s)$

$\Longrightarrow \left\{ \begin{array}{l} A\left(u_{\$(m)}\right)^2 = \varphi\left(u_{\$(m)}\right) b_{\$(m)}\left|A\left(u_{\$(m)}\right)\right|^2 \\ \lim b_{\$(m)} = \frac{B(s_1)}{A(s_1)} = b \in IR_* \text{ exists} \end{array} \right.$

$\Longrightarrow \lim b_{\$(m)} \neq 0$

otherwise $B(s_1) = 0 \Rightarrow A(s_1) = \varphi(s_1)B(s_1) = 0$ absurd!!, because $A(s_1) \in iIR_*$.

So $\lim A\left(u_{\$(m)}\right)^2 = \lim\left[\varphi\left(u_{\$(m)}\right) b_{\$(m)}\left|A\left(u_{\$(m)}\right)\right|^2\right]$

$\Longrightarrow A(s_1)^2 = \varphi(s_1)\,b\,|A(s_s)|^2 \in IR^-_* \Longrightarrow \varphi(s_1) \in IR_*$ ■

**Theorem 21** *If the Riemann conjecture is true so,*

$\left\{ \begin{array}{l} (i): \mathrm{Re}[S(s)] \in IR^-_* \quad \forall s \in \mathbb{S} \text{ such } \mathrm{Re}(s) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[ \\ (ii): S(s) = g(s)S(1-s) \quad \forall s \in \mathbb{S} \text{ such } \mathrm{Re}(s) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[ \\ (iii): S\left(\frac{1}{2} + ic\right) \in IR^- \end{array} \right.$

Proof. For $(i)$ : First, we have see in the last lemme: If the Riemann conjecture is true so, $\varphi(s) \in IR\ \forall s \in \mathbb{S}$ such $\mathrm{Re}(s) \in \left]0, \frac{1}{2}\right[$.

$\Longrightarrow S(s) = g(s)S(1-s)$

Also the line $\operatorname{Im}(s) = 0$, $s = r \in ]0,1[$ and since $\frac{1}{n^r} < \frac{1}{(n-1)^r}$ $\forall r \in ]0,1[$

$$S(s) = S(r) = \sum_{n\geq 1} \frac{(-1)^n}{n^r} = -1 + \frac{1}{2^r} + -\frac{1}{3^r} + \frac{1}{4^r} + \ldots < 0.$$

So, $\operatorname{Re}[S(s)] \in IR^-_* \ \forall s \in$ s such $\operatorname{Re}(s) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[$ $(i)$

because, otherwise $\exists s \in$ s/ $\operatorname{Re}(s) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[$ such $\operatorname{Re}[S(s)] \in IR^+$, $\operatorname{Re}[S(s)] \in IR^-_* \Longrightarrow \exists s_0 \in ]r, s]$ such $\operatorname{Re}[S(s_0)] = 0$ (the intermediate value theorem), with $]r, s] = \{r + ic \in$ s$/0 < c \leq \operatorname{Im}(s)\}$ and $r = \operatorname{Re}(s)$.

For $(ii)$ : As $\operatorname{Re}[S(s)] \in IR^-_* \ \forall s \in$ s such $\operatorname{Re}(s) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[$ $(i)$

so, also $\operatorname{Re}[S(1-s)] \in IR^-_*$

$\Longrightarrow S(s) = g(s) S(1-s) \ \forall s \in$ s such $\operatorname{Re}(s) \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[$ $(ii)$

since $S(s) = g(s) S(1-s)$

For $(iii)$: As we have $(i)$ :

$\operatorname{Re}[S(r+ic)] \in IR^-_* \ \forall c \in IR$ such $r \in \left]0, \frac{1}{2}\right[ \cup \left]\frac{1}{2}, 1\right[$, so $\lim_{r\to\frac{1}{2}} \operatorname{Re}[S(r+ic)] \in IR^-$

$\Longrightarrow S\left(\frac{1}{2} + ic\right) \in IR^-$. ■

Theorem 22 *(Sorry!! but): fhe Riemann conjecture is not true.*

Proof. Let $E = \left\{\frac{1}{2} + ic/c \in IR\right\}$, we have: If The Riemann conjecture is true, so $S\left(\frac{1}{2} + ic\right) \in IR^-$

$\Longrightarrow \forall s \in E$, $S(s) = \overline{S(s)}$

$S(s)$ and $\overline{S(s)}$ are analytic and holomorphic functions: Because, if $f$ satisfies the Cauchy-Riemann equations then $\overline{f}$ also satisfies it (Using the Cauchy-Riemann equations and Schwarz's theorem).

$\overline{S(s)}$ and $S(s)$ are two analytic functions take the same values on $E$, so $S(s)$ and $\overline{S(s)}$ take the same values on s

$\Longrightarrow S(s) \in IR^- \ \forall s \in$ s.

Absurd!! ■

# 5 Conclusions

While conventionally accepted, our current investigation indicates a potential refutation of the Riemann Hypothesis. Contrary to the widely accepted conjecture, our analysis establishes that the Riemann Hypothesis is false. And this was made possible thanks to our last two publications, in which we made these announcements and new conjectures equivalent to the Riemann Hypothesis. The first conjecture announced: In the band $s$ $(s = r + ic)$ a complex such that its real part is strictly between 0 and 1 $(0 < r < 1)$, we have the real part of the Dirchlet function ($\operatorname{Re}[S(s)] = 0$) can only be zero in the straight line "the real part of $s$ is equal to 0.5" $(r = \frac{1}{2})$. So instead of studying and finding the zeros of the Dirchlet function ($S(s)$ or $\eta(s)$), we just need to study its real part which is equivalent to studying and finding the zeros of the real function $R' = \sum_{n=1}^{+\infty} (-1)^n \frac{\cos[\ln(n)c]}{n^r}$ with two variables $(r, c)$. While the second conjecture informs us about what the zeros can be $(8_{\frac{}{2}}{}^{1} + ic \ = (2k+1)\,\pi)$ in the straight line $r = \frac{1}{2}$.

This refutation challenges a century-old conjecture and necessitates a re-evaluation of current theoretical frameworks concerning the distribution of prime numbers.

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