Remarks on vanishing elements of a finite group

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Abstract

Let G be a finite group, and let $g \in G$. We say that the element g is a vanishing element in G if there exists an irreducible character χ of G such that $\chi(g) = 0$. In this paper, we establish a number of results on the vanishing elements of a finite group.

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1 Introduction and Preliminary

Throughout this paper, the term group always means a group of finite order, and by simple groups we mean nonabelian simple groups. The letter G always denotes a group, and $\pi(G)$ denotes the set of all prime divisors of the order $|G|$ of a group G. For an element $x \in G$, $o(x)$ denotes the order of x. In addition, we use also the following notation:

$\pi_e(G) = \{o(x) | x \in G\}$, and $\pi_e^*(G) = \pi_e(G) - \{1\}$.
$Van(G) = \{x \in G |$ there exists $\chi \in Irr(G)$ such that $\chi(x) = 0\}$.
$Vo(G) = \{o(x) | x \in Van(G)\}$.
$\Gamma(G)$: The vanishing prime graph of G(see [2]).
$V(\Gamma(G))$: The set of vertices of $\Gamma(G)$.
$n(\Gamma(G))$: The number of connected components of $\Gamma(G)$).
$GK(G)$: The prime graph of G(the Gruenberg-Kegel graph of G) (see [3]).
$n(GK(G))$: The number of connected components of $GK(G)$.

If $GK(G)$ is disconnected, we denote by $\pi_i(G)$ the i'th connected component of $GK(G)$, where $i = 1, 2, \cdots, n(GK(G))$, and we suppose that $2 \in \pi_1(G)$ if 2 is a vertex of $GK(G)$.

Let N be a set of positive integers. We put $N|_O = \{x | x \in N$ and x is odd $\}$ and $N|_2 = \{x | x \in N, x > 1$ and x is a power of $2\}$. Assume that $2 \in N$. Then we say that the period of 2 in N is m if $2^m = Max(N|_2)$.

All further unexplained notation is standard and is referred to [1], for example.

Let $g \in G$. We say that the element g is a vanishing element of G if there exists an irreducible character χ of G such that $\chi(g) = 0$. Clearly, $Van(G)$ is the set of vanishing elements of G. By a classical theorem of W.Burnside, if G is a nonabelian group, then $Van(G)$ is not empty (see [1, 6.13, p.76]). Hence, if G is a nonabelian group, then the set $Vo(G)$ of orders of vanishing elements of G is not empty. The set $Vo(G)$ encodes non-trivial information about the structure of G. Therefore, in [4], the following conjecture was put forward.

Conjecture A: *Let S be a simple group. If $|G| = |S|$ and $Vo(G) = Vo(S)$, then $G \cong S$.*

Clearly, confirming this conjecture is an interesting topic.

We define the V-recognition of a group G as follows. For an arbitrary subset v of the set of positive integers ≥ 2, we denote by $h(v)$ the number of pairwise non-isomorphic groups G such that $Vo(G) = v$. Given a group G, G is said to be V-recognizable if $h(Vo(G)) = 1$, almost V-recognizable if $1 < h(Vo(G)) < \infty$, and non-V-recognizable if $h(Vo(G)) = \infty$.

Clearly, the following Problem B is also interesting.

Problem B: Which simple groups are V-recognizable?

In this paper, we establish a number of results related to vanishing elements or Conjecture A, and we establish several results on Problem B

In order to complete the proofs of results of the present paper, we first list several lemmas which will be used in the sequel.

Lemma 1.1[5, Corollary A]. *Assume that the order of every vanishing element of G can not be divided by a prime p. Then G has a normal Sylow p-subgroup.*

Let p be a prime divisor of $|G|$, and let $\chi \in Irr(G)$. We say that χ is of p-defect zero if p does not divide $|G|/\chi(1)$. If $\chi \in Irr(G)$ is of p-defect zero, then, for every element $g \in G$ such that p divides $o(g)$, we have $\chi(g) = 0$ (see [6, (8.17), p.133]).

Lemma 1.2[7, Corollary 2]. *Let S be a simple group and assume that there exists a prime q such that S does not have an irreducible character of q-defect zero. Then* $q = 2$ *or* 3 *and S is isomorphic to one of the following groups:* $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ *and* A_n *with* $n \geq 7$.

From Lemma 1.2 we get the following:

Lemma 1.3. *Let S be a simple group, and assume that S is not isomorphic any one of the following groups:* $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ *and* A_n *with* $n \geq 7$. *Then* $Van(S) = S - \{1\}$ *and* $Vo(S) = \pi^*(S)$.

Lemma 1.4[8, LEMMA 2.7]. *Let N be a normal subgroup of G, and let p be a prime divisor of* $|N|$. *If N has an irreducible character of p-defect zero, then every element of N of order divisible by p is a vanishing element of G.*

Lemma 1.5[8, THEOREM B]. *Let G be a nonsolvable group. If* $\Gamma(G)$ *is disconnected, then G has a unique nonabelian compose factor S. Moreover* $n(\Gamma(G)) \leq n(GK(S))$ *unless G is isomorphic to* A_7.

Lemma 1.6[8, Proposition 2.10]. *Let S be a sporadic simple group, or an alternating group on n letters with* $n \geq 8$. *Then S has an irreducible character* φ *which extends to* $Aut(S)$ *and an element g of order* 6 *such that* $\varphi(g) = 0$.

Lemma 1.7[2]. *Let G be a solvable group. Then* $n(\Gamma(G)) \leq 2$. *Further, if* $n(\Gamma(G)) = 2$, *then two connected components of* $\Gamma(G)$ *are complete graphs.*

Let N be a normal subgroup of G. It is well known that we can identity the irreducible characters of G/N with the irreducible characters of G that contains N in the kernel. So, it is obvious that the following Lemma 1.8 holds.

Lemma 1.8[2, Remark 2.2]. *Let N be a normal subgroup of a group G. The following statements are true:*
(i) *If* $xN \in Van(G/N)$, *then* $xN \subseteq Van(G)$.
(ii) *Each element in* $Vo(G/N)$ *is a factor of some element in* $Vo(G)$.

Lemma 1.9(see [8, Proposition 4.2]). *Assume that* $V(\Gamma(G)) \neq \pi(G)$. *Then* $\Gamma(G)$ *is connected.*

Lemma 1.10[8, Proposition 6.4]. *Let S be a simple group. Then* $\Gamma(S) = GK(S)$, *unless* $S \cong A_7$.

2 A result on vanishing prime graphs and its consequences

By the definition of the vanishing prime graph $\Gamma(G)$, we have

$V(\Gamma(G)) = \{p|\ p$ **is a prime and there exists an element** $m \in Vo(G)$ **such that** $p|m\}$.

For two distinct vertices $p, q \in V(\Gamma(G))$, p and q are adjacent in $\Gamma(G)$ if and only if there exists an element $m \in Vo(G)$ such that $pq|m$.

In this section, we first establish a theorem on a group G for which the vertex 2 of the vanishing prime graph $\Gamma(G)$ is an isolated vertex, and then we establish several consequences of this theorem, namely, Theorem 2.5 in the present section.

Suppose that $|V(\Gamma(G))| \geq 2$ and $2 \in V(\Gamma(G))$. Clearly, 2 is an isolated vertex of $\Gamma(G)$ if and only if each element of $Vo(G)$ is either an odd number or a power of 2.

Proposition 2.1. *Let G be a nonsolvable group. Then $|V(\Gamma(G))| \geq 3$ and $2 \in V(\Gamma(G))$.*

Proof. Suppose on the contrary that $2 \notin V(\Gamma(G))$, that is, the order of every vanishing element of G is not divisible by 2. Then by Lemma 1.1 G has a normal Sylow 2-subgroup, and thus by the Schur-Zassenhaus theorem and the odd order theorem we conclude that G is solvable, contradicting the hypothesis. So, $2 \in V(\Gamma(G))$. If $|V(\Gamma(G))| \leq 2$, then by Lemma 1.1 and the Burnside $\{p, q\}$-theorem we conclude that G is solvable, contradicting the hypothesis. So, we have $|V(\Gamma(G))| \geq 3$. This completes the proof. $\Box$

The following Proposition 2.2 is obvious.

Proposition 2.2. *Let H be a group. Assume that H has a normal 2-subgroup P, and the order of each element of H is either an odd number or a power of 2. Then the order of each element of H/P is either an odd number or a power of 2 and* $\pi_e(H)|_O = \pi_e(H/P)|_O$.

Proposition 2.3. *Let G be a nonsolvable group, and suppose that the order of each element of G is either an odd number or a power of 2. Then $G/O_2(G)$ is isomorphic to one of the following groups: $L_2(q)$, $q = 2^k$ with $k \geq 2$ or q is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{n+1})$, $n \geq 1$; $L_3(4)$; $A_6.2_3$ (using the notation in the Atlas[9]).*

Proof. Since the order of each non-identity element of G is either an odd number or a power of 2, $C_G(t)$ is a 2-group for every involution t of G. Then, noting that G is nonsolvable, by [10, III, Theorem 5] we conclude that $G/O_2(G)$ is isomorphic to one of the following groups: $L_2(q)$, $q = 2^k$ with $k \geq 2$ or q is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{n+1})$, $n \geq 1$; $L_3(4)$; $A_6.2_3$. This completes the proof. $\Box$

Theorem 2.4. *Let G be a nonsolvable group, and let K be the maximal solvable normal subgroup of G. Assume that $\Gamma(G)$ is disconnected. Then G has a normal series $K < M \leq G$ such that M/K is a simple group, $G/K \leq Aut(M/K)$ and $G/M \leq Out(M/K)$. Furthermore, one of the following statements holds:*

(1) $M/K \cong A_n$ with $n \geq 5$. If $n \neq 6$, then $G/K \cong A_n$ or S_n. If $n = 6$, then G/K is isomorphic one of the following groups: A_6, S_6, $PGL(2, 6)$ and $A_6.2_3$. Furthermore, if $n \geq 8$, then $6 \in Vo(G/K)$.

(2) M/K is a simple group of Lie type, and $\pi^*(M/K) = Vo(M/K) \subseteq Vo(G/K)$.

(3) M/K is a sporadic simple group, and $6 \in Vo(G/K)$.

Proof. Since G is nonsolvable and $\Gamma(G)$ is disconnected by the hypothesis, by Lemma 1.5 G has a normal series

$$1 \leq K < M \leq G$$

such that M/K is a simple group, $G/K \leq Aut(M/K)$ and $G/M \leq Out(M/K)$.

Since M/K is a simple group, by the classification of finite simple groups we conclude that M/K is isomorphic to either an alternating group A_n with $n \geq 5$, or a simple group of Lie type, or a sporadic simple group.

(i) Assume that $M/K \cong A_n$ with $n \geq 5$.

If $n \neq 6$, then $Aut(A_n) = S_n$. Then, since $M/K \leq G/K \leq Aut(M/K)$ and $|S_n : A_n| = 2$, we have that $G/K \cong A_n$ or S_n. If $n = 6$, by checking in the Atlas [9] we conclude that G/K is isomorphic one of the following groups: A_6, S_6, $PGL(2, 6)$ and $A_6.2_3$.

If $n \geq 8$, by Lemma 1.6 we conclude that $6 \in Vo(G/K)$. So, (1) holds.

(ii) Assume that M/K is a simple group of Lie type.

By Lemma 1.2, Lemma 1.3 and Lemma 1.4, we have that $\pi^*(M/K) = Vo(M/K) \subseteq Vo(G/K)$. So, (2) holds.

(iii) Assume that M/K is a sporadic simple group.

Since $G/K \leq Aut(M/K)$, by Lemma 1.6 we conclude that $6 \in Vo(G/K)$. So, (3) holds, and the proof of the theorem is completed. $\Box$

In the proof of the following Theorem 2.5, we shall use the following fact: Let G be a simple group of Lie type over the field $GF(2^n)$. Then $Out(G)$ is a cyclic group of order n.

Theorem 2.5. *Let G be a nonsolvable group, and let K be the maximal solvable normal subgroup of G. Assume that 2 is an isolated vertex of $\Gamma(G)$. Then G has a normal series $K < M \leq G$ such that M/K is a simple group, $G/K \leq Aut(M/K)$ and $G/M \leq Out(M/K)$. Furthermore, the following propositions (1), (2), (3) and (4) hold:*

(1) M/K is isomorphic to one of the following groups: A_7; $L_2(q)$, $q = 2^n$ with $n \geq 2$ or q is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{2n+1})$, $n \geq 1$; $L_3(4)$.

(2) Assume that $K > 1$, and that every non-identity element of G/K is vanishing in G/K. Let V be a normal subgroup of G such that $V < K$ and K/V is a chief factor of G. Set $\widetilde{G} = G/V$. Then the following statements hold:

(2a) Each element of $\pi^*_e(\widetilde{G})$ is either an odd number or a power of 2.

(2b) $\widetilde{K}$ ($= K/V$) is an elementary abelian 2-group.

(2c) Let $p \in \pi(K)$ such that $K \neq O^p(K)$. Then $p = 2$. In particular, if K is nilpotent, then $K = O_2(G)$.

(2d) If each element in $\pi^*_e(\widetilde{G})$ is a prime power, then $\widetilde{G}/\widetilde{K}$ is isomorphic to one of the following groups: A_5, $L_2(8)$, $Sz(2^3)$ and $Sz(2^5)$.

(2e) $\pi_e(\widetilde{G})|_O = \pi_e(G/K)|_O$.

(2f) $Max(\pi_e(\widetilde{G})|_2) \leq Max(Vo(G)|_2)$ and $Max(\pi_e(\widetilde{G})|_2) \geq Max(\pi_e(G/K)|_2)$.

(3) The following statements hold:

(3a) Assume that $M/K \cong L_2(2^n)$ with $n \geq 2$. Then $G/K = M/K$. Furthermore, if the period of 2 in $Vo(G)$ is 1, then $K = 1$ and $G \cong L_2(2^n)$.

(3b) Assume that $M/K \cong Sz(2^{2n+1})$ with $n \geq 1$. Then $G/K = M/K$. Furthermore, if the period of 2 in $Vo(G)$ is 2, then $K = 1$ and $G \cong Sz(2^{2n+1})$.

(3c) Assume that $M/K \cong A_7$. Then $G/K \cong A_7$.

(3d) Assume that $M/K \cong L_2(7)$. Then $G \cong L_2(7)$.

(3e) Assume that $M/K \cong L_2(17)$. Then $G \cong L_2(17)$.

(3f) Assume that $M/K \cong L_3(4)$. Then $G \cong L_3(4)$.

(3g) Assume that $M/K \cong L_2(9)$. Then $G \cong L_2(9)$ or $G \cong A_6.2_3$.

(4) If the period of 2 in $Vo(G)$ is 1, then $G \cong L_2(2^n)$ with $n \geq 2$.

Proof. Write $\overline{G} = G/K$. Since 2 is an isolated vertex of $\Gamma(G)$ by the hypothesis, each element in $Vo(G)$ is either an odd number or a power of 2, that is, the order of every vanishing element of G is either an odd number or a power of 2. Hence, by Lemma 1.8 we conclude that each element in $Vo(\overline{G})$ is either an odd number or a power of 2.

Since G is nonsolvable, by Proposition 2.1 we have that $|V(\Gamma(G))| \geq 3$. Then, since 2 is an isolated vertex of $\Gamma(G)$, $\Gamma(G)$ is disconnected. It follows by Theorem 2.4 that G has a normal series $K < M \leq G$ such that $\overline{M}$ ($= M/K$) is a simple group, $\overline{G} \leq Aut(\overline{M})$ and $G/M \leq Out(M/K)$.

Notice that $A_5 \cong L_2(2^2)$ and $A_6 \cong L_2(3^2)$. Then, since each element in $Vo(\overline{G})$ is either an odd number or a power of 2, by Theorem 2.4 we conclude that either $\overline{M} \cong A_7$, or $\overline{M}$ is isomorphic to a simple group of Lie type and the order of every non-identity element of $\overline{M}$ is either an odd number or a power of 2. Hence, by Proposition 2.3 we conclude that either $\overline{M} \cong A_7$, or $\overline{M}$ is isomorphic to one of the following groups: $L_2(q)$, $q = 2^n$ with $n \geq 2$ or q is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{n+1})$, $n \geq 1$; $L_3(4)$. So, (1) holds.

Next, we prove (2). By the assumption of (2), we have $\widetilde{G} = G/V$. Clearly, $\widetilde{K}$ ($= K/V$) is an elementary abelian p-group, where p is a prime. We have that $\widetilde{G}/\widetilde{K} = G/V/K/V \cong G/K$. Then, by the assumption of (2) we conclude that every non-identity element of $\widetilde{G}/\widetilde{K}$ is vanishing in $\widetilde{G}/\widetilde{K}$. Hence, by Lemma 1.8 we have that

$$(*)\ \ \widetilde{G} - \widetilde{K} \subseteq Van(\widetilde{G}), \text{ and } \pi^*_e(\widetilde{G}) \subseteq Vo(\widetilde{G}) \cup \{p\}.$$

In addition, by Lemma 1.8 every element of $Vo(\widetilde{G})$ is either an odd number or a power of 2. It follows that every element in $\pi^*_e(\widetilde{G})$ is either an odd number or a power of 2. So, (2a) holds.

Since every element in $\pi_e^*(\widetilde{G})$ is either an odd number or a power of 2. by Proposition 2.3 $\widetilde{G}$ has a normal 2-subgroup $\widetilde{U}$ such that $\widetilde{G}/\widetilde{U}$ is isomorphic to one of the following groups: $L_2(q)$, $q = 2^n$ with $n \geq 2$ or q is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{n+1})$, $n \geq 1$; $L_3(4)$; $A_6.2_3$. Then we conclude that $\widetilde{U} = \widetilde{K}$. Hence, $p = 2$ and $\widetilde{K}(= K/V)$ is an elementary abelian 2-group. So, (2b) hold. By (2b) we conclude that (2c) holds.

Assume that each element in $\pi_e(\widetilde{G})$ is a prime power, then by [11, Theorem 1.7] we have that $\widetilde{G}/O_2(\widetilde{G})$ is isomorphic to one of the following groups: A_5, $L_2(8)$, $Sz(2^3)$ and $Sz(2^5)$. By (2b) it is obvious that $\widetilde{K} = O_2(\widetilde{G})$. So, (2d) holds.

Since K is a normal 2-subgroup of G and every element of $\pi_e(\widetilde{G})$ is either an odd number or a power of 2 (see (2a) and (2b)), by Proposition 2.2 we have $\pi_e(\widetilde{G})|_O = \pi_e(\widetilde{G}/\widetilde{K})|_O = \pi_e(G/K)|_O$. So, (2e) holds.

Let $\widetilde{x} \in \widetilde{G}$ be a non-identity element such that $o(\widetilde{x})$ is a power of 2. By Lemma 1.8 every element of $Vo(\widetilde{G})$ is a factor of some element of $Vo(G)$. Then, noting that $p = 2$, by $(*)$ we conclude that $o(\widetilde{x})$ is a factor of some element of $Vo(G)$, and thus $Max(\pi_e(\widetilde{G})|2) \leq Max(Vo(G)|_2)$. Since $\widetilde{G}/\widetilde{K} \cong G/K$, it is obvious that $Max(\pi_e(\widetilde{G})|2) \geq Max(\pi_e(G/K)|_2)$. So,(2f) holds. This completes the proof of (2).

Below, we prove (3). Set $\overline{G} = G/K$.

(i) Assume that $\overline{M} = M/K \cong L_2(2^k)$, where $k \geq 2$.

$Out(\overline{M})(= Out(L_2(2^k)))$ is a cyclic group of order k. Then, since $\overline{G}/\overline{M} \cong G/M \leq Out(\overline{M})$, $\overline{G}/\overline{M}$ is a cyclic group of order m, where $m|k$.

We will show that $|\overline{G} : \overline{M}|$ is a power of 2(including the case when $\overline{G} = \overline{M}$). Suppose on the contrary that $|\overline{G} : \overline{M}|$ is not a power of 2. Then there exists a normal subgroup R of G such that $M < R \leq G$ and $|\overline{R}/\overline{M}| = r$(if $k = 2n + 1$, take $R = G$), where r is an odd number. $\overline{M}$ has an unique irreducible character(Steinberg character) χ such that $\chi(1) = |\overline{M}|_2$ (see [12, Theorem 38.1,p.228]). Clearly, χ is invariant in $\overline{G}$. It follows by [6, (11.22), p.186] that χ extends to $\overline{R}$. So, there exists $\varphi \in Irr(\overline{R})$ such that φ is of 2-defect zero, and thus by [6, (8.17), p.133] we conclude that every element of order $2s$ in $\overline{R}$ is a vanishing element of $\overline{R}$, where s is an odd prime. Then by Lemma 1.4 we conclude that every element of order $2s$ in $\overline{R}$ is a vanishing element of $\overline{G}$. Then, since each element in $Vo(\overline{G})$ is either an odd number or a power of 2, $\overline{R}$ does not have any element of order $2s$, where s is an odd prime. Then by Proposition 2.3 $\overline{R}$ has a normal 2-subgroup $\overline{U}$ such that $\overline{R}/\overline{U}$ is a simple group, and thus $r = 2$, a contradiction. Hence, $|\overline{G} : \overline{M}|$ is a power of 2, that is, $|G/K : M/K|$ is a power of 2. Let r be any odd prime divisor of $|\overline{G}|$. Then r is a prime divisor of $|\overline{M}|$ because $|\overline{G} : \overline{M}|$ is a power of 2. By Lemma 1.2 $\overline{M}$ has an irreducible character ϑ of r-defect zero. Let ψ be an irreducible constituent of $\vartheta^{\overline{G}}$. Then $\psi \in Irr(\overline{G})$ is of r-defect zero, and thus $\psi(g) = 0$ for any element of order $2r$ in $\overline{G}$. Then, since each element in $Vo(\overline{G})$ is either an odd number or a power of 2, $\overline{G}$ does not have elements of order $2r$, where r is any odd prime divisor of $|\overline{G}|$. Then, noting that $\overline{M} \cong L_2(2^k)$, by Proposition 2.3 we conclude that $\overline{G}/O_2(\overline{G})$ is a simple group. Hence, since K is the maximal solvable normal subgroup of G, we have that $O_2(\overline{G}) = O_2(G/K) = 1$ and $\overline{G} = \overline{M}$, that is, $G/K = M/K$. So, the first conclusion of (3a) holds.

Now, we assume that the period of 2 in $Vo(G)$ is 1, that is, $Max(Vo(G)|_2) = 2$. We will show that $K = 1$. Suppose on contrary that $K > 1$, and let V be a normal subgroup of G such that $V < K$ and K/V is a chief factor of G. Set $\widetilde{G} = G/V$. We have that $L_2(2^k) \cong M/K = G/K$. It follows by [13, 8.27, p.213] that

$$\pi_e(G/K) = \{1, 2, \text{all factors of } 2^k - 1 \text{ and } 2^k + 1\}.$$

By Lemma 1.3, every non-identity element of G/K is vanishing in G/K. Hence, by (2) we get that

$$\pi_e(G/V) = \pi_e(\widetilde{G}) = \{1, 2, \text{all factors of } 2^k - 1 \text{ and } 2^k + 1\} = \pi_e(L_2(2^k)).$$

Then by [14] we have that $G/V \cong L_2(2^k)$. On the other hand, we have that $G/K \cong L_2(2^k)$. It follows that $|G/K| = |G/V|$ and $K = V$, a contradiction. Hence, we have that $K = 1$ and $G \cong L_2(2^k)$. So, the second conclusion of (3a) holds. This completes the proof of (3a).

(ii) Assume that $M/K = \overline{M} \cong Sz(2^{2n+1})$.

$Out(M/K)(= Out(Sz(2^{n+1}))$ is a cyclic group of order $2n + 1$. It follows that $G/\overline{M}$ ($\cong G/M \leq Out(M/K) = Out(\overline{M})$) is a cyclic group of odd order. $\overline{M}$ has a unique irreducible character χ such that $\chi(1) = |\overline{M}|_2$ (see [15, Chap. XI, Theorem 5.10, p.216]). Then by using the same argument as in the third paragraph of (i)(G replaces R), we conclude that $G/K = \overline{G} = \overline{M} = M/K$. So, the first conclusion of (3b) is true.

Now, we assume that the period of 2 in $Va(G)$ is 2, that is, $Max(Vo(G)|_2) = 4$. We have that $\pi_e(\overline{G}) = \pi_e(\overline{M}) = \pi_e(Sz(2^{2n+1})) = \{1, 2, 4,$ all factors of $(2^{2n+1} - 1), (2^{2n+1} - 2^{n+1} + 1)$ and $(2^{2n+1} + 2^{n+1} + 1)\}$ (see [16]). By using the same argument as in the final paragraph of (i) and by [16], we conclude that $K = 1$ and $G = M = Sz(2^{2n+})$. So, the second conclusion of (3b) is true. Then (3b) holds.

(iii) Assume that $M/K = \overline{M} \cong A_7$.

By Theorem 2.4 either $G/K \cong A_7$ or $G/K \cong S_7$. Since each element $Vo(G/K)$ is either an odd number or a power of 2, by checking in the Atlas [9] we conclude that $G/K \cong A_7$, that is, (3c) holds.

(iv) Assume that $\overline{M} = M/K \cong L_2(7)$.

We have that $\overline{G} \leq Aut(\overline{M}) = Aut(L_2(7))$. Since the order of every element in $Vo(\overline{G})$ is either an odd number or a power of 2, by checking in the Atlas [9] we conclude that $G/K = \overline{G} = \overline{M} \cong L_2(7)$. Note that $Vo(G/K) = Vo(L_2(7)) = \pi^*_e(L_2(7) = \{2, 4, 3, 7\}$ (see [6, p.289]).

We will show that $K = 1$. Suppose on the contrary that $K > 1$. Let V be a normal subgroup of G such that $V < K$ and K/V is a chief factor of G. Set $\tilde{G} = G/V$. Since $G/K \cong L_2(7)$, by Lemma 1.3 every non-identity element of G/K is vanishing in G/K, and so we can apply (2). By (2) we have that each element in $\pi_e(\tilde{G})$ is either an odd number or a power of 2, and $\pi_e(\tilde{G})|_O = \pi_e(G/K)|_O = \pi_e(L_2(7))|_O = \{3, 7\}$. Hence, each element in $\pi_e(\tilde{G})$ is a prime power. Then by (2) we conclude that $\tilde{G}/\tilde{K} \ncong L_2(7)$(see (2d)). Then, since $\tilde{G}/\tilde{K} \cong G/K$, $G/K \ncong L_2(7)$, a contradiction. So, we have that $K = 1$ and $G \cong L_2(7)$, that is, (3d) holds.

(v) By using the same argument as in (iv) we conclude that (3e) and (3f) hold.

(vi) Assume that $\overline{M} = M/K \cong L_2(9)$.

By Theorem 2.4 we have that G/K is isomorphic to one of the following groups: $A_6, S_6, PGL(2, 9)$ and $A_6.2_3$. Since each element in $Vo(G/K)$ is either an odd number or a power of 2, by checking in the Atlas[9] it is easy to see that either $G/K \cong A_6$ or $G/K \cong A_6.2_3$.

Assume that $G/K \cong A_6$ ($\cong L_2(9)$). By using the same argument in the final paragraph of (iv), we conclude that $K = 1$ and $G \cong A_6$

Assume that $G/K \cong A_6.2_3$. Note that every non-identity element of $A_6.2_3$ is vanishing in $A_6.2_3$ (see the Atlas[9]), and so G/K satisfies the assumption of (2). Then by using the same argument in the final paragraph of (iv), we conclude that $K = 1$ and $G \cong A_6.2_3$. Then we have proved that (3g) holds. This completes the proof of (3).

Finally, we prove (4). Then we assume that the period of 2 in $Vo(G)$ is 1. We have proved that either $\overline{M} \cong A_7$, or $\overline{M}$ is isomorphic to one of the following groups: $L_2(q), q = 2^k$ or q is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{n+1}), n \geq 1; L_3(4)$.

Suppose that $M/K = \overline{M} \cong A_7$. By (3) we have that $G/K \cong A_7$. By checking in Atlas[9],we have that $4 \in Vo(G/K)$, and thus by Lemma 1.8 we get that the period of 2 in $Vo(G)$ is greater than 1, a contradiction. Hence, $M/K \ncong A_7$. It follows that M/K is isomorphic to a simple group of Lie type. Then by Lemma 1.2, Lemma 1.3 and Lemma 1.4, we conclude that every non-identity element of M/K is a vanishing element of G/K. Then, if a Sylow 2-subgroup of M/K is not elementary abelian, then G/K has a vanishing element of order 4, and so by Lemma 1.8 we conclude that the period of 2 in $Vo(G)$ is greater than 1, a contradiction. So, we conclude that a Sylow 2-subgroup of M/K is an elementary abelian 2-group, and thus $M/K \cong L_2(2^k)$ with $k \geq 2$. Then by (3) we have that $G \cong L_2(2^k)$, that is, (4) holds. This completes the proof of the theorem. $\Box$

Corollary 2.6[17, Main Theorem]. *Assume that $Vo(G) = Vo(L_2(2^a))$ with $a \geq 2$. Then $G = L_2(2^a)$.*

Proof. Let K be the maximal solvable normal subgroup of G. By the hypothesis, Lemma 1.3 and [13, 8.27, p.213], we have that $Vo(G) = Vo(L_2(2^a)) = \pi^*_e(L_2(2^a)) = \{2,$ all factors of $2^a - 1$ and $2^a + 1\} - \{1\}$. Hence, $\Gamma(G)$ is disconnected and $n(\Gamma(G)) = 3$. Then

by Lemma 1.7 we conclude that G is nonsolvable. Clearly, 2 is an isolated vertex in $\Gamma(G)$, and the period of 2 in $Vo(G)$ is 1. It follows by Theorem 2.5(4) that $G \cong L_2(2^k)$. Then by Lemma 1.3 we have that $\pi_e^*(G) = Vo(G) = Vo(L_2(2^a)) = \pi_e^*(L_2(2^a))$, and thus $\pi_e(G) = \pi_e(L_2(2^a))$. Hence, by [14] we conclude that $G \cong L_2(2^a)$. This completes the proof. *2*

By Corollary 2.6, the simple group $L_2(2^a)$ with $a \geq 2$ is V-recognizable.

Theorem 2.7. *Let G be a nonsolvable group, and let K be the maximal solvable normal subgroup of G. If $3 \notin V(\Gamma(G))$ and 2 is an isolated vertex of $\Gamma(G)$, then $G/K \cong Sz(2^{2n+1})$ with $n \geq 1$.*

Proof. Put $\overline{G} = G/K$. By the hypothesis and Theorem 2.5, G has a normal subgroup M such that $K \leq M$, $\overline{M} = M/K$ is a simple group. By the hypothesis we have that $3 \notin V(\Gamma(G))$. Then by Lemma 1.1 we have that $3 \nmid |\overline{M}|$. A simple group S with $3 \nmid |S|$ is isomorphic to $Sz(2^{n+1})$ with $n \geq 1$(see [15, 3.7 Remarks, p.188]). Hence, $\overline{M} \cong Sz(2^{2n+1})$ with $n \geq 1$. Then by Theorem 2.5(3) we conclude that $G/K \cong Sz(2^{2n+1})$ with $n \geq 1$. This completes the proof of the theorem. *2*

Theorem 2.8. *Let G be a nonsolvable group. Assume that G satisfies the following three conditions: (i) $3 \notin V(\Gamma(G))$, (ii) 2 is an isolated vertex of $\Gamma(G)$, and (iii) The period of 2 in $Vo(G)$ is 2. Then $G \cong Sz(2^{2n+1})$ with $n \geq 1$.*

Proof. Let K be the maximal solvable normal subgroup of G. By Theorem 2.7 we have that $G/K \cong Sz(2^{2n+1})$, where $n \geq 1$. Then, since the period of 2 in $Vo(G)$ is 2, by Theorem 2.5(3) we conclude that $G \cong Sz(2^{2n+1})$. This completes the proof of the theorem. *2*

Corollary 2.9[18, Main Theorem]. *If $Vo(G) = Vo(Sz(2^{2n+1}))$, where $n \geq 1$, then $G \cong Sz(2^{2n+1})$.*

Proof. Since $Sz(2^{2n+1})$ is a simple group of Lie type, by Lemma 1.3 we have that $Vo(Sz(2^{2n+1})) = \pi_e^*(Sz(2^{2n+1}))$. We have that $\pi_e(Sz(2^{2n+1})) = \{1, 2, 4$, all factors of $(2^{2n+1}-1)$ and $(2^{2n+1} - 2^{n+1} + 1)$, and $(2^{2n+1} + 2^{n+1} + 1)\}$(see [16]). In addition, 3 does not divide $|Sz(2^{2n+1})|$(see [15, 3.7 Remarks, p.188]). Then, since $Vo(G) = Vo(Sz(2^{2n+1}))$ by the hypothesis, we conclude that 2 is an isolated vertex in $\Gamma(G)$, $3 \notin V(\Gamma(G))$ and the period of 2 in $OV(G)$ is 2. It follows by Theorem 2.8 that $G \cong Sz(2^{2m+1})$, where $m \geq 1$. Then we have that $Vo(Sz(2^{2m+1})) = Vo(G) = Vo(Sz(2^{2m+1}))$. On the other hand, we have that $Vo(Sz(2^{2m+1})) = \pi_e^*(Sz(2^{2m+1}))$ and $Vo(Sz(2^{2n+1})) = \pi_e^*(Sz(2^{2n+1}))$. Hence, we have that $\pi_e(Sz(2^{2n+1})) = \pi_e(Sz(2^{2m+1})) = \pi_e(G)$, and thus $G \cong Sz(2^{2n+1})$ (see [16]). This completes the proof.

By Corollary 2.9, the simple group $Sz(2^{2n+1})$ is V-recognizable.

By Theorem 2.5 we get the following.

Corollary 2.11. *Let G be a nonsolvable group. The following two propositions hold:*
(1) If the period of 2 in $Vo(G)$ is 1 and $\Gamma(G) = \Gamma(L_2(2^n))$, where $n \geq 2$, then $G \cong L_2(2^n)$.
(2) If the period of 2 in $Vo(G)$ is 2 and $\Gamma(G) = \Gamma(Sz(2^{2n+1}))$, then $G \cong Suz(2^{2n+1})$.

The following theorem is an improvement of [19,Theorem 1.1].

Theorem 2.12. *Assume that G is nonsolvable and every element in $Vo(G)$ is a prime power. Then the following propositions (1),(2) and (3) hold:*

(1) If $O_2(G) = 1$, then G is isomorphic to one of the following groups: A_7, A_5, $L_2(7)$, $L_2(8)$, $L_2(9)$, $L_2(17)$, $L_3(4)$, $Sz(8)$, $Sz(32)$ and $A_6.2_3$.

(2) If $O_2(G) \neq 1$, then one of the following holds:

(2a) The period of 2 in $Vo(G)$ is greater than 1 and $G = [N]A$, where $A \cong A_5 \cong SL_2(4)$ and $N (= O_2(G))$ is the direct product of minimal normal subgroups of G, each of which is of order 2^4 and as a G/N-module is isomorphic to the natural $GF(2^2)SL_2(2^2)$-module.(We denote by $[A]B$ the split extension of its normal subgroup A by a complement B.)

(2b) The period of 2 in $Vo(G)$ is greater that 1, $G/O_2(G) \cong L_2(8)$, and $O_2(G)$ is the direct product of minimal normal subgroups of G, each of which is of order 2^6 and as a $G/O_2(G)$-module is isomorphic to the natural $GF(2^3)SL_2(2^3)$-module.

(2c) The period of 2 in $Vo(G)$ is greater than 2, $G/O_2(G) \cong Sz(2^3)$, and $O_2(G)$ is the direct product of minimal normal subgroups of G, each of which is of order 2^{12} and as a $G/O_2(G)$-module is isomorphic to the natural $GF(2^3)Sz(2^3)$-module of dimension 4.

(2d) The period of 2 in $Vo(G)$ is greater than 2, $G/O_2(G) \cong Sz(2^5)$, and $O_2(G)$ is the direct product of minimal normal subgroups of G, each of which is of order 2^{20} and as a $G/O_2(G)$-module is isomorphic to the natural $GF(2^5)Sz(2^5)$-module of dimension 4.

(3) If the period of 2 in $Vo(G)$ is 1, then $G \cong A_5$ or $L_2(8)$.

Proof. Let K be the maximal solvable normal subgroup of G. By the hypothesis and Proposition 2.1,G is nonsolvable, $2 \in V(\Gamma(G))$, and 2 is an isolated vertex in $\Gamma(G)$.So, we can apply Theorem 2.5.

By Theorem 2.5, G has a normal series $K < M \leq G$ such that M/K is a simple group, $G/K \leq Aut(M/K)$ and $G/M \leq Out(M/K)$. Furthermore, M/K is isomorphic to one of the following groups: A_7; $L_2(q)$, $q = 2^n$ with $n \geq 2$ or q is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{2n+1})$, $n \geq 1$; $L_3(4)$. Hence, either $M/K \cong A_7$ or M/K is a simple group of Lie type.

Next, we show that if $K > 1$, then K is a 2-group. Suppose that $K > 1$ and K is not a 2-group. Then G has a normal series $1 \leq T < R \leq K \leq G$ such that R/T is a chief factor of G of odd order. Suppose $T \neq 1$. Considering the group G/T, by induction we may assume that K/T is a 2-group, and so R/T is a 2-group, a contradiction. Hence, $T = 1$ and R is an elementary abelian r-group, where r is an odd prime. Considering the group G/R, by induction we may assume that K/R is a 2-group. It follows that $K = RP$, where P is a 2-group. By Burnside $\{p, q\}$-theorem, G has an s-element g, where s is a prime with $r \neq s \neq 2$. Assume that $M/K \cong A_7$. By Theorem 2.5(3) we have that $G/K = M/K \cong A_7$. Noting that $\pi(A_7) = \{2, 3, 5, 7\}$, $Vo(A_7) = \{2, 3, 4, 5, 7\}$ and $\pi^*(A_7) = \{2, 3, 4, 5, 6, 7\}$(see the Atlas[9]), by Lemma 1.8 we conclude that $gK \subseteq Van(G)$. Assume that M/K is a simple group of Lie type. Then by Lemma 1.3 and Lemma 1.8 we have that $gK \subseteq Van(G)$)(We may assume that $g \in M$). So, in any case, we have that $gK \subseteq Van(G)$. Then, since every element in $Vo(G)$ is a prime power, $< g >$ acts fixed-point freely on $K = RP$, and so K is nilpotent. Hence, by Theorem 2.5(2) K is a 2-group, a contradiction. So, K is a 2-group and $K = O_2(G)$.

We already know that either $G/K \cong A_7$ or M/K is a simple group of Lie type. We discuss two cases separately as follows..

(I) $G/K \cong A_7$

Suppose $K > 1$. Then $K = O_2(G)$ and $G/O_2(G) \cong A_7$. We have that $Vo(G/O_2(G)) = Vo(A_7) = \{2, 3, 4, 5, 7\}$ and $\pi_e(G/O_2(G)) = \pi_e(A_7) = \{1, 2, 3, 4, 5, 6, 7\}$. A_7 has an irreducible character of 3-defect zero (se the Atlas[9]), and so evert element in G whose order is divisible by 3 is vanishing in G. Hence, letting x be any 3-element of G, $xO_2(G) \in Van(G/O_2(G))$, and so by Lemma 1.8 we have that $xO_2(G) \subseteq Van(G)$. It follows that the order of every element in $xO_2(G)$ is a prime power. Hence, letting P be a Sylow 3-subgroup of G, P acts fixed-point freely on $O_2(G)$, and thus P is a cyclic group, contradicting the fact that a Sylow 3-subgroup of A_7 is an elementary abelia group of order 9. So, $K = 1$ and $G \cong A_7$.

(II) M/K **is a simple of Lie type.**

In this case, by Lemma 1.2, Lemma 1.4 and Lemma 1.8, we conclude that every element of $\pi_e^*(M/K)$ is a prime power, and so M/K is isomorphic to one of the following groups $A_5 (\cong L_2(4))$, $L_2(7)$, $L_2(8)$, $L_2(9)(\cong A_6)$, $L_2(17)$, $Sz(8)$ and $Su(32)$ (see [11, Theorem 1.7]). It follows from Theorem 2.5(3) that either (1) holds or one of the following cases occurs:

(i) $O_2(G) \neq 1$**, the period of 2 in $Vo(G)$ is greater than 1, and $G/O_2(G) \cong A_5 (\cong L_2(4))$.**

(ii) $O_2(G) \neq 1$**, the period of 2 in $Vo(G)$ is greater than 1, and $G/O_2(G) \cong L_2(8)$.**

(iii) $O_2(G) \neq 1$**, the period of 2 in $Vo(G)$ is greater than 2, and $G/O_2(G) \cong Sz(2^3)$.**

(iv) $O_2(G) \neq 1$**, the period of 2 in $Vo(G)$ is greater than 2, and $G/O_2(G) \cong Sz(2^5)$.**

In addition, by Lemma 1.3 and Lemma 1.8 we conclude that $G - O_2(G) \subseteq Vo(G)$, and so every element in $\pi_e(G)$ is a prime power. Then by [11,Theorem 1.7] we conclude that one of (2b),(2c) and (2d) hold. Now, we assume that $G/O_2(G) \cong A_5 (\cong L_2(4))$. Then we have that $|G| = 2^m \cdot 3 \cdot 5$. Let $x \in G$ be of order 3. By Lemma 1.3 and Lemma 1.8 we have that $xO_2(G) \subseteq Van(G)$ and $x^2O_2(G) \subseteq Van(G)$. It follows that $< x >$ acts point-fixed freely point on $O_2(G)$. Then it is obvious that $C_G(< x >) =< x >$, and thus by [11, Theorem

1.7] and [20, Theorem] we conclude that $G = [N]A$**, where** $A \cong A_5 \cong L_2(4)$ **and** $N (= O_2(G))$ **is the direct product of minimal normal subgroups of** G**, each of which is of order** 2^4 **and as a** G/N **-module is isomorphic to the natural** $GF(2^2)SL_2(2^2)$**-module. Furthermore, by Theorem 2.5(3) we have that the period of** 2 **in** $Vo(G)$ **is greater than** 1**. So, (2a) holds. Then (2) holds.**

By Theorem 2.5(4), (3) holds. This completes the proof of the theorem. $\Box$.

3 Three basic theorems

The following three theorems are useful for the investigation of Conjecture A and Problem B.

Theorem 3.1. *Let S be a simple group with* $S \ncong A_7$*, and assume that* $GK(S)$ *is disconnected and* $n(GK(S)) \geq 3$*. Assume that* $Vo(G) = Vo(S)$*. Then G has a normal series* $K < M \leq G$ *such that K is the maximal solvable normal subgroup of G, M/K is a simple group and* $G/K \leq Aut(M/K)$*. Moreover,* $\pi(G) = \pi(S)$*,* $\Gamma(G) = \Gamma(S) = GK(S)$ *and* $n(GK(S)) \leq n(GK(M/K))$*. In addition, the following two statements hold:*

(1) If S is not isomorphic to any one of the following groups: $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ *and* A_n *with* $n \geq 7$*, then* $Vo(G/K) \subseteq \pi_e(S)$*.*

(2) If S and M/K are not isomorphic to any one of the following groups: $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ *and* A_n *with* $n \geq 7$*, then* $\pi_e(M/K) \subseteq \pi_e(S)$*.*

Proof. Since S **is a simple group and** $S \ncong A_7$**, by Lemma 1.10 we have that** $\Gamma(S) = GK(S)$**. Then, since** $Vo(G) = Vo(S)$**, we have that** $\Gamma(G) = \Gamma(S) = GK(S)$**. It follows by the hypothesis that** $\Gamma(G)$ **is disconnected and** $n(\Gamma(G)) \geq 3$**. Then by Lemma 1.7 we conclude that** G **is nonsolvable.** $G \ncong A_7$**; otherwise,** $Vo(A_7) = Vo(G) = Vo(S)$**, and thus by Theorem 2.5 we have that** $S \cong A_7$ **(see also [19, Theorem 1.4]), contradicting the hypothesis. It follows by Theorem 2.4 and Lemma 1.5 that** G **has a normal series** $K < M \leq G$ **such that** K **is the maximal solvable normal subgroup of** G**,** M/K **is a simple group,** $G/K \leq Aut(M/K)$**, and** $n(GK(S)) \leq n(GK(M/K))$**. By Lemma 1.9 we have that** $\pi(G) = V(\Gamma(G)) = V(\Gamma(S)) = \pi(S)$**. Hence,** $\pi(G) = \pi(S)$**.**

Assume that S **is not isomorphic to any one of the following groups:** $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ **and** A_n **with** $n \geq 7$**. Then by Lemma 1.3 we have that** $Vo(G) = Vo(S) = \pi_e^*(S)$**. By Lemma 1.8 we have that each element of** $Vo(G/K)$ **is a factor of some element of** $Vo(G)$**. Then each element of** $Vo(G/K)$ **is a factor of some element of** $\pi_e^*(S)$**, and so** $Vo(G/K) \subseteq \pi_e^*(S)$**, that is, (1) holds.**

Assume that S **and** M/K **are not isomorphic to any one of the following groups:** $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ **and** A_n **with** $n \geq 7$**. Then by Lemma 1.3 we have that** $Vo(M/K) = \pi_e^*(M/K)$ **and** $Vo(S) = \pi_e^*(S)$**. Furthermore, by Lemma 1.2 and Lemma 1.4 we conclude that** $Vo(M/K) \subseteq Vo(G/K)$**. Hence, by (1) we have that** $\pi_e^*(M/K) = Vo(M/K) \subseteq Vo(G/K) \subseteq \pi_e^*(S)$**. Then,** $\pi_e(M/K) \subseteq \pi_e(S)$**, that is, (2) holds. This completes the proof of the theorem.** $\Box$

Theorem 3.2. *Let S be a simple group. Assume that* $n(GK(S)) = 2$ *and there exists a connected component* ρ *of* $GK(S)$ *such that* ρ *is not a complete graph. Suppose that* $Vo(G) = Vo(S)$*. Then G has a normal series* $K < M \leq G$ *such that K is the maximal solvable normal subgroup of G, M/K is a simple group and* $G/K \leq Aut(M/K)$*. Moreover,* $\pi(G) = \pi(S)$*,* $\Gamma(G) = \Gamma(S) = GK(S)$ *and* $n(GK(S)) \leq n(GK(M/K))$*. In addition, the following two statements hold:*

(1) If S is not isomorphic to any one of the following groups: $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ *and* A_n *with* $n \geq 7$*, then* $Vo(G/K) \subseteq \pi_e(S)$*.*

(2) If S and M/K are not isomorphic to any one of the following groups: $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM$ *and* A_n *with* $n \geq 7$*, then* $\pi_e(M/K) \subseteq \pi_e(S)$*.*

Proof. Since $n(GK(S)) = 2$ **by the hypothesis, we have that** $S \ncong A_7$ **because** $n(GK(A_7)) = 3$ **(see the Atlas[9]). Hence by Lemma 1.10 we have that** $\Gamma(S) = GK(S)$**. Then, since** $Vo(G) = Vo(S)$ **by the hypothesis, we have that** $\Gamma(G) = \Gamma(S) = GK(S)$**, and so by the hypothesis we have that** $n(\Gamma(G)) = 2$ **and** $\Gamma(G)$ **has a connected component** ρ **such that** ρ **is not a complete graph. Hence, by Lemma 1.7 we know that** G **is nonsolvable.** $G \ncong A_7$**; otherwise,** $Vo(G) = Vo(A_7) = \{2, 3, 4, 5, 7\}$ **and** $n(\Gamma(G)) = 4$**, a contradiction. Then, by using**

the same argument as in the proof of Theorem 3.1 we conclude that the theorem holds. 2

Theorem 3.3. *Let S be a simple group, and assume that S satisfies the following two conditions: (i) S is not isomorphic any one of the following groups: M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_2, BM and A_n with $n \geq 7$; (ii) If $\pi_e(G) = \pi_e(S)$, then $G \cong S$. Then the following proposition (*) holds:*

(*) Assume that $\pi(G) = \pi(S)$, $Vo(G) = Vo(S)$ and $G/K \cong S$, where K is the maximal solvable normal subgroup of G, then $K = 1$ and $G \cong S$.

Proof. Suppose that $K > 1$. Let V be a normal subgroup of G such that $V < K$ and K/V is a chief factor of G. Set $\tilde{G} = G/V$. Clearly, $\tilde{G}/\tilde{K} \cong G/K \cong S$, and $\tilde{K}\,(= K/V)$ is an elementary abelian p-group, where $p \in \pi(G) = \pi(S)$. By the hypothesis and Lemma 1.3 every non-identity of S is a vanishing element of S, that is, $\pi_e^*(S) = Vo(S)$. Then every non-identity of $\tilde{G}/\tilde{K}$ ($\cong S$) is a vanishing element in $\tilde{G}/\tilde{K}$, and so by Lemma 1.8 we have that

$$\tilde{G} - \tilde{K} \subseteq Van(\tilde{G}).$$

It follows that $\pi_e^*(\tilde{G}) \subseteq Vo(\tilde{G}) \cup \{p\}$, where $p \in \pi(G) = \pi(S)$. By Lemma 1.8, every element in $Vo(\tilde{G})$ is a factor of some element in $Vo(G)(= Vo(S) = \pi_e^*(S))$. Hence, $\pi_e^*(\tilde{G}) \subseteq \pi_e^*(S))$. On the other hand, since $\tilde{G}/\tilde{K} \cong S$, we have that $\pi_e^*(S) \subseteq \pi_e^*(\tilde{G})$. Therefore, we get that $\pi_e(\tilde{G}) = \pi_e(S)$. Then by the hypothesis we have that $G/V = \tilde{G} \cong S$. Then, since $G/K \cong S$, we get that $V = K$, a contradiction. So, $K = 1$ and $G \cong S$. The proof is finished. 2.

4 Several results related to Conjecture A

In this section, we will use theorems 3.1, 3.2 and 3.3 to establish several results related to Conjecture A. For this, we first give a table about simple K_3-groups. Let S be a simple group. If $|\pi(S)| = n$, then S is called a simple K_n-group. If S is a simple K_3-group, then S is isomorphic to one of the following groups: A_5 ($\cong L_2(2^2)$), A_6 ($\cong L_2(3^2)$), $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$ (see [21, p.12]). By checking in the Atlas [9], we obtain the following Table 1.

Table 1 Simple K_3-groups

G	$\|G\|$	$\pi_e^*(G) = Vo(G)$	$n(\Gamma(G)) = n(GK(G))$
A_5	$2^2 \cdot 3 \cdot 5$	{2, 3, 5}	3
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	{2, 3, 4, 7}	3
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	{2, 3, 7, 9}	3
$L_2(17)$	$2^3 \cdot 3^2 \cdot 17$	{2, 4, 8, 3, 9, 17}	3
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	{2, 3, 4, 6, 8, 13}	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	{2, 3, 4, 6, 7, 8, 12}	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	{2, 3, 4, 5, 6, 9, 12}	2
A_6	$2^3 \cdot 3^2 \cdot 5$	{2, 3, 4, 5}	3

Let $p_1, \cdots, p_r$ be distinct primes, and let $|G| = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \cdot n$, where n is a $\{p_1, \cdots, p_r\}'$-number. We write $|G|_{\{p_1,\cdots,p_r\}} = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $|G|_{p_1} = p_1^{\alpha_1}$.

Theorem 4.1. *Assume that $|G|_{\{3,5\}} = |L_2(31)|_{\{3,5\}}$ and $Vo(G) = Vo(L_2(31))$. Then $G \cong L_2(31)$.*

Proof. We have that $|L_2(31)| = 2^5{\cdot}3{\cdot}5{\cdot}31$ and $\pi_e^*(L_2(31)) = Vo(L_2(31)) = \{2, 3, 4, 5, 8, 15, 16, 31\}$ (see the Atlas [9]). Clearly, $GK(L_2(31))$ has three connected components: $\pi_1 = \{2\}$, $\pi_2 = \{3, 5\}$ and $\pi_3 = \{31\}$. Since $Vo(G) = Vo(L_2(31))$ by the hypothesis, by Theorem 3.1 we conclude that G has a normal series $K < M \leq G$ such that K is the maximal solvable normal subgroup of G, M/K is a simple group and $G/K \leq Aut(M/K)$. Moreover, $\pi(G) = \pi(L_2(31)) = \{2, 3, 5, 31\}$, $\Gamma(G) = \Gamma(L_2(31)) = GK(L_2(31))$ and $n(GK(M/K)) \geq 3$. Notice that 2 is an isolated vertex of $\Gamma(G)(= GK(L_2(31))$, and so we can use Theorem 2.5.

Clearly, $\pi(M/K) \subseteq \{2, 3, 5, 31\}$. Then either $|\pi(M/K)| = 3$ or $\pi(M/K) = \{2, 3, 5, 31\}$. We discuss the two cases separately as follows.

(I) $|\pi(M/K)| = 3$.

In this case, M/K **is a simple** K_3**-group. By the hypothesis we have that** $|G|_{\{3,5\}} = |L_2(31)|_{\{3,5\}} = 3 \cdot 5$**. Then, noting that** $n(GK(M/K)) \geq 3$ **and** $\pi(M/K) \subseteq \{2, 3, 5, 31\}$**, by Table 1 we conclude that** $M/K \cong A_5 (\cong L_2(4))$**. Hence by Theorem 2.5(3) we have that** $G/K \cong A_5$**.It follows that** $|G/K| = 2^2 \cdot 3 \cdot 5$ **and** $\pi(K) \subseteq \{2, 31\}$**. Let** x **be any element of** G **of order** 3**. By Lemma 1.3 and Lemma 1.8 we have that** $xK \subseteq Vo(G)(= \{2, 3, 4, 5, 8, 15, 16, 31\})$**. Then, since** 2 **and** 31 **are isolated vertices of** $\Gamma(G)(= GK(L_2(31)))$**,** $< x >$ **acts fixed-point freely on** K**, and thus** K **is nilpotent. Hence, by Theorem 2.5(2) we get that** $K = O_2(G)$**. Then we have that** $\pi(G) = \pi(A_5) = \{2, 3, 5\}$**, contradicting the fact that** $31 \in \pi(G)$**.**

(II) $\pi(M/K) = \{2, 3, 5, 31\}$.

In this case, by Table 1 in [22] we have that either $M/K \cong L_2(31)$ **or** $M/K \cong L_3(5)$**. If** $M/K \cong L_3(5)$**, then** $|M/K| = 2^5 \cdot 3 \cdot 5^2 \cdot 31$**(see the Atlas[9]), and** $3 \cdot 5 = |G|_{\{3,5\}} \geq |M/K|_{\{3,5\}} = 3 \cdot 5^2$**, a contradiction. Therefore, we have that** $M/K \cong L_2(31)$**. Since** $G/K \leq Aut(M/K)$**, either** $G/K \cong L_2(31)$ **or** $G/K \cong L_2(31).2$ **(see the Atlas[9]). Suppose that** $G/K \cong L_2(31).2$**. Then** $6 \in Vo(G/K)$ **(see the Atlas[9]), and so by Lemma 1.8 we conclude that** 6 **is a factor of some element in** $Vo(G)(= \{2, 3, 4, 5, 8, 15, 16, 31\})$**, a contradiction. Hence,** $G/K \cong L_2(31)$**. We know that,for a group** H**, if** $\pi_e(H) = \pi_e(L_2(31))$**, then** $H \cong L_2(31)$ **(see [23, Theorem 2.7]). Therefore, by Theorem 3.3 we conclude that** $K = 1$ **and** $G \cong L_2(31)$**. This completes the proof of the theorem.** 2

By Table 1 in [22] and by checking in the Atlas[9], we get the following Table 2.

Table 2 Simple groups G **with** $\pi(G) = \{2, 3, 5, 11\}$**.**

G	$\|G\|$	$Vo(G)$	$V(\Gamma(G))$	$n(\Gamma(G))$
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	{2, 3, 5, 6, 11}	{2, 3, 5, 11}	3
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	{2, 3, 4, 5, 6, 8, 11}	{2, 3, 5, 11}	3
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	{2, 3, 4, 5, 6, 8, 10, 11}	{2, 3, 5, 11}	2
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	{2, 3, 4, 5, 6, 8, 9, 11, 12, 15, 18}	{2, 3, 5, 11}	2

Theorem 4.2. *The following two propositions hold:*
(1) Assume that $|G|_{11} = |L_2(11)|_{11}$ *and* $Vo(G) = Vo(L_2(11))$*. Then* $G \cong L_2(11)$*;*
(2) Assume that $|G|_{\{2,3\}} = |L_2(11)|_{\{2,3\}}$ *and* $Vo(G) = Vo(L_2(11))$*. Then* $G \cong L_2(11)$*.*

Proof. The proof of (1): We have that $|L_2(11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$ **and** $\pi_e^*(L_2(11)) = Vo(L_2(11) = \{2, 3, 5, 6, 11\}$ **(see Table 2 and Lemma 1.3). Clearly,** $GK(L_2(11)$ **has three connected components:** $\{2, 3\}$**,** $\{5\}$ **and** $\{11\}$**. By the hypothesis we have that** $Vo(G) = Vo(L_2(11))$**. Then by Theorem 3.1 we conclude that** G **has a normal series** $K < M \leq G$ **such that** K **is the maximal solvable normal subgroup of** G**,** M/K **is a simple group, and** $G/K \leq Aut(M/K)$**. Moreover,** $\pi(G) = \pi(L_2(11)) = \{2, 3, 5, 11\}$**,** $\Gamma(G) = \Gamma(L_2(11)) = GK(L_2(11)$ **and** $n(GK(M/K)) \geq 3$**. It follows that** $\pi(M/K) \subseteq \{2, 3, 5, 11\}$**, and** M/K **is either a simple** K_3**-group or a simple** K_4**-group.**

(I) Assume M/K **is a simple** K_3**-group.**

Since $\pi(M/K) \subseteq \{2, 3, 5, 11\}$ **and** $n(GK(M/K)) \geq 3$**, by Table 1 we have that** $M/K \cong A_5$ **or** A_6**.**

(Ia) Assume that $M/K \cong A_5$**.**

Since $G/K \leq Aut(M/K)$**,** $G/K \cong A_5$ **or** S_5**. If** $G/K \cong S_5$**, then** $10 \in Vo(G/K)$ **(see the Atlas[9]), and so by Theorem 3.1(1) we have that** $10 \in \pi^*(L_2(11)) = \{2, 3, 5, 6, 11\}$**, a contradiction. So, we have that** $G/K \cong A_5 (\cong L_2(4))$**. It follows that** $|G/K| = 2^2 \cdot 3 \cdot 5$**. Then, since** $\pi(G) = \{2, 3, 5, 11\}$ **and** $|G|_{11} = |L_2(11)|_{11} = 11$ **by the hypothesis, we have that** $\pi(K) \subseteq \{2, 3, 11\}$ **and** $|K|_{11} = 11$**. Let** P **be a Sylow** 11**-subgroup of** K**. We have that** $|P| = 11$**. By Frattini argument we have that** $G = KN_G(P)$**, and so** $G/K \cong N_G(P)/N_K(P)$**. It follows that there exists a** 3**-element** $x \in G - K$ **such that** $< x > \leq N_G(P)$**. Then, since** $|Aut(P)| = 10$**, we have that** $[< x >, P] = 1$**, and so** xP **contains an element of order** $k \cdot 33$**. By Lemma 1.3 and Lemma 1.8 we have that** $xP \subseteq xK \subseteq Van(G)$**. It follows that** $k \cdot 33 \in Vo(G) = Vo(L_2(11)) = \{2, 3, 4, 5, 6, 7, 8, 9, 12\}$**, a contradiction.**

(Ib) Assume that $M/K \cong A_6 (\cong L_2(9))$**.**

We have that $|M/K| = |A_6| = 2^3 \cdot 3^2 \cdot 5$**,** $G/K \leq Aut(M/K) = Aut(A_6)$**. Then** $|G/K| = 2^3 \cdot 3^2 \cdot 5$ **or** $2^4 \cdot 3^2 \cdot 5$**(see the Atlas[9]), Hence, by using the argument used in (Ia) we will get a contradiction.**

(II) Assume M/K is a simple K_4-group.

In this case, $\pi(M/K) = \{2, 3, 5, 11\}$. Since $n(GK(M/K)) \geq 3$, by Table 2 we conclude that $M/K \cong M_{11}$ or $L_2(11)$. Then by Theorem 3.1(2) and Table 2 we conclude that $M/K \cong L_2(11)$. Since $G/K \leq Aut(M/K) = Aut(L_2(11))$, we have that $G/K \cong L_2(11)$ or $L_2(11).2$ (see the Atlas[9]). If $G/K \cong L_2(11).2$, then $10 \in Vo(G/K)$, and so by Theorem 3.1(1) we have that $10 \in \pi^*_e(L_2(11)) = \{2, 3, 5, 6, 11\}$, a contradiction. Therefore, we have that $G/K \cong L_2(11)$. For a group H, if $\pi_e(H) = \pi_e(L_2(11))$, then $H \cong L_2(11)$ (see Table 1 in [24]). So, by Theorem 3.3 we conclude that $K = 1$ and $G \cong L_2(11)$. This completes the proof of (1).

The proof of (2) is left to the reader. The proof of the theorem is finished. $\Box$

By using the same argument as in the proof of Theorem 4.2 we conclude that the following theorem holds:

Theorem 4.3. Assume that $|G|_{\{3,11\}} = |M_{11}|_{\{3,11\}}$ and $Vo(G) = Vo(M_{11})$. Then $G \cong M_{11}$.

By Theorem 3.2 and by using the argument used in the proof of Theorem 4.2, we conclude that the following theorem holds.

Theorem 4.4. Let S be a simple group which is isomorphic to M_{12} or $U_5(2)$. Assume that $|G|_{\{3,11\}} = |S|_{\{3,11\}}$ and $Vo(G) = Vo(S)$. Then $G \cong S$.

By Table 1 in [22] and by checking in the Atlas [9], we obtain the following Table 3.

Table 3 Simple groups G with $\pi(G) = \{2, 3, 5, 13\}$.

G	$\lvert G\rvert$	$Vo(G) = \pi^*_e(G)$	$n(\Gamma(G) = n(GK(G))$
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$\{2, 3, 4, 5, 6, 12, 13\}$	3
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$\{2, 3, 4, 5, 10, 13, 15\}$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	$\{2, 3, 4, 5, 6, 8, 10, 12, 13, 20\}$	2
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	$\{2, 3, 4, 5, 6, 10, 12, 13, 15, 20, 30\}$	2
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	$\{2, 3, 4, 5, 6, 8, 10, 12, 13, 16\}$	2

Theorem 4.5. *Assume that* $|G|_{13} = |L_2(25)|_{13}$ *and* $Vo(G) = Vo(L_2(25))$. *Then* $G \cong L_2(25)$.

Proof. $|L_2(25)| = 2^3 \cdot 3 \cdot 5^2 \cdot 13$ and $\pi^*_e(L_2(25)) = Vo(L_2(25)) = \{2, 3, 4, 5, 6, 12, 13\}$ (see Table 3). Clearly, $GK(L_2(25)))$ has three connected components: $\{2, 3\}$, $\{5\}$ and $\{13\}$. By the hypothesis we have that $Vo(G) = Vo(L_2(25)) = \{2, 3, 4, 5, 6, 12, 13\}$. Then by Theorem 3.1 G has a normal series $K < M \leq G$ such that K is the maximal solvable normal subgroup of G, M/K is a simple group and $G/K \leq Aut(M/K)$. Moreover, $\pi(G) = \pi(L_2(25)) = \{2, 3, 5, 13\}$, $\Gamma(G) = \Gamma(L_2(25)) = GK(L_2(25)$ and $n(GK(M/K)) \geq 3$. It follows that $\pi(M/K) \subseteq \{2, 3, 5, 31\}$. Then $|\pi(M/K)| = 3$ or 4. We discuss the two cases separately as follows.

(I) Assume that $|\pi(M/K)| = 3$.

In this case, M/K is a simple K_3-group. Then, since $\pi(M/K) \subseteq \{2, 3, 5, 13\}$ nd $n(GK(M/K) \geq 3$, by Table 1 we conclude that $M/K \cong A_5$ or A_6.

(Ia) Assume that $M/K \cong A_5$ ($\cong L_2(4)$).

Since $G/K \leq Aut(M/K) = Aut(A_5) = S_5$, we have that either $G/K = M/K \cong A_5$ or $G/K \cong S_5$. If $G/K \cong S_5$, then $10 \in Vo(G/K)$(see the Atlas[9]), and so by Lemma 1.8 we conclude that 10 is a factor of some element in $Vo(G) = \{2, 3, 4, 5, 6, 12, 13\}$, a contradiction. Hence, we have that $G/K \cong A_5$. Then $|G/K| = 2^2 \cdot 3 \cdot 5$ and $13 \in \pi(K)$. Let R be a Sylow 13-subgroup of K. By the hypothesis we have that $|R| = |G|_{13} = |L_2(25)|_{13} = 13$. In view of Frattini argument, we have that $G = KN_G(R)$, and so $G/K \cong N_G(R)/N_K(R)$. Then there exists a 5-element $x \in G - K$ such that $x \in N_G(R)$. We have that $|Aut(R)| = 13 - 1 = 12$. Then, since x is a 5-element, we have that $[<x>, R] = 1$, and so xR contains an element of order $k \cdot 65$. Clearly, xK is an element of $G/K (\cong A_5)$ of order 5. By Lemma 1.3 and Lemma 1.8, we have that $xK \subseteq Van(G)$. It follows that $k \cdot 65 \in Vo(G) = \{2, 3, 4, 5, 6, 12, 13\}$, a contradiction.

(Ib) Assume that $M/K \cong A_6$.

Note that $A_6 \cong L_2(9)$, $|Out(A_6)| = 4$ and $Vo(A_6) = \pi^*_e(A_6) = \{2, 3, 4, 5\}$ (see Table 1). Since $G/K \leq Aut(M/K) = Aut(A_6)$, by checking in Atlas[9] we conclude that $|G/K| = 2^3 \cdot 3^2 \cdot 5$ or $2^4 \cdot 3^2 \cdot 5$. Notice that every element of G/K of order 5 is vanishing in G/K(see the Atlas [9]). So, by using the same argument as in (Ia) we will get a contradiction.

(II) **Assume that** $\pi(M/K) = \{2, 3, 5, 13\}$**.**

Noting that $n(GK(M/K)) \geq 3$**, by Table 3 we conclude that** $M/K \cong L_2(25)$**. Then, since** $G/K \leq Aut(M/K)$**, by checking in the Atlas [9] we conclude that** $G/K = M/K \cong L_2(25)$**. We know that, for a group** H**, if** $\pi_e(H) = \pi_e(L_2(25))$**, then** $H \cong L_2(25)$ **(see [24, Table 1]). Therefore, by Theorem 3.3 we conclude that** $K = 1$ **and** $G \cong L_2(25)$**. This completes the proof of the theorem.** 2

By using Theorem 3.2 and by using the same argument as in the proof of Theorem 4.5, we conclude that the following theorem holds.

Theorem 4.6. *Assume that* $|G|_{13} = |U_3(4)|_{13}$ *and* $Vo(G) = Vo(U_3(4))$*. Then* $G \cong U_3(4)$*.*

We have that $|L_2(19)| = 19(19-1)(19+1)/2 = 2^2 \cdot 3^2 \cdot 5 \cdot 19$ **and** $\pi_e^*(L_2(19)) = Vo(L_2(19)) = \{2, 3, 5, 9, 10, 19\}$ **(see the Atlas[9]). Clearly,** $GK((L_2(19))$ **has three connected components:** $\{2, 5\}$**,** $\{3\}$ **and** $\{19\}$**. In addition, if** S **is a simple group with** $\pi(S) = \{2, 3, 5, 19\}$ **and** $|S|_{\{5,19\}} = 5 \cdot 19$**, then** $S \cong L_2(19)$ **(see Table 1 in [22]). So, by using the same argument as in the proof of Theorem 4.5, we can prove that the following theorem holds.**

Theorem 4.7. *Assume that* $|G|_{\{5,19\}} = |L_2(19)|_{\{5,19\}}$ *and* $Vo(G) = Vo(L_2(19))$*. Then* $G \cong L_2(19)$.

By checking in the Atlas[9], we get that $|L_3(8)| = 2^5 \cdot 3^2 \cdot 7^2 \cdot 73$ **and** $\pi_e^*(L_3(8)) = Vo(L_3(8)) = \{2, 3, 7, 9, 14, 21, 63, 73\}$**. Clearly,** $GK(L_3(8))$ **has two connected components:** $\pi_1 = \{2, 3, 7\}$ **and** $\pi_2 = \{73\}$**, and** π_1 **is not a complete graph. In addition, if** S **is a simple group with** $\pi(S) = \{2, 3, 7, 73\}$**, then** $S \cong L_3(8)$ **(see Table 1 in [22]). According to these information, by Theorem 3.2 we conclude that the following theorem holds.**

Theorem 4.8. *Assume that* $|G|_{73} = |L_3(8)|_{73}$ *and* $Vo(G) = Vo(L_3(8))$*. Then* $G \cong L_3(8)$*.*

Theorem 4.9. *Assume that* $|G|_{\{17,19\}} = |J_3|_{\{17,19\}}$ *and* $Vo(G) = Vo(J_3)$*. Then* $G \cong J_3$*.*

Proof. $|J_3| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$**, and** $GK(J_3)$ **has three connected components:** $\pi_1 = \{2, 3, 5\}$, $\pi_2 = \{17\}$ **and** $\pi_3 = \{19\}$ **(see the Atlas[9]). By the hypothesis we have that** $Vo(G) = Vo(J_3)$**. Hence, by Theorem 3.1** G **has a normal series** $K < M \leq G$ **such that** K **is the maximal solvable normal subgroup of** G**,** M/K **is a simple group and** $G/K \leq Aut(M/K)$**. Moreover,** $\pi(G) = \pi(J_3) = 2 \cdot 3 \cdot 5 \cdot 17 \cdot 19$**,** $\Gamma(G) = \Gamma(J_3) = GK(J_3)$ **and** $n(GK(M/K)) \geq 3$**. It follows that** $\pi(M/K) \subseteq \{2, 3, 5, 17, 19\}$**, and** $|\pi(M/K)| = 3, 4$ **or** 5**.**

(I) Assume that $|\pi(M/K)| = 3$**.**

By Table 1 we conclude that $M/K \cong A_5, A_6$ **or** $L_2(17)$**. By using the same argument as in (I) of the proof of Theorem 4.5, we will get a contradiction.**

(II) Assume that $\pi(M/K)| = 4$**.**

By Table 1 in [22] we have that $M/K \cong S_4(4)$ **or** $L_2(19)$**.**

Assume that $M/K \cong S_4(4)$**. Then** $|M/K| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$**(see the Atlas[9]). Since** $G/K \leq Aut(M/K)$**, we have that** $|G/K| = 2^n \cdot 3^2 \cdot 5^2 \cdot 17$**, where** $n = 8, 10$ **or** 12**(see the Atlas[9]). Then, since by the hypothesis** $|G|_{\{17,19\}} = |J_3|_{\{17,19\}} = 17 \cdot 19$**, we have that** $\pi(K) \subseteq \{2, 3, 5, 19\}$ **and** $|K|_{19} = 19$**. Let** P **be a Sylow** 19**-subgroup of** K**. Then we have that** $|P| = 19$**. Let** $x \in M$ **be of order** 17**. Since** $K \not\subset G$ **and** $(|K|, 17) = 1$**, we may assume that** $< x > \leq N_G(P)$**, and so** $[< x >, P] = 1$**. By Lemma 1.2,Lemma 1.3, Lemma 1.4 and Lemma 1.8, we have that** $xP \subseteq xK \subseteq Van(G)$**. It follows that** 17 **and** 19 **are adjacent in** $\Gamma(G)(= GK(J_3))$**, a contradiction.**

Assume $M/K \cong L_2(19)$**. Since** $G/K \leq Aut(M/K) = Aut(L_2(19))$**, we have that** $|G/K| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$ **or** $2^3 \cdot 3^2 \cdot 5 \cdot 19$ **(see the Atlas[9]). Then, since** $|G|_{\{17,19\}} = 17 \cdot 19$**, we have that** $(|K|, 19) = 1$ **and** $|K|_{17} = 17$**. Thus, by using the same argument as in the above paragraph, we will get a contradiction.**

(III) Assume that $|\pi(M/K|)| = 5$**.**

In this case, $\pi(M/K|) = \{2, 3, 5, 17, 19\}$**. By Table 1 in [22] we get that** $M/K \cong J_3$**. Then, since** $G/K \leq Aut(M/K)$**, by checking in Atlas[9] we conclude that** $G/K \cong M/K \cong J_3$**. We know that, for a group** H**, if** $\pi_e(H) = \pi_e(J_3)$**, then** $H \cong J_3$ **(see [23, Theorem 2.7]). Therefore, by Theorem 3.3 we conclude that** $K = 1$ **and** $G \cong J_3$**. This completes the proof of the theorem.** 2

Theorem 4.10 *Assume that* $|G|_{19} = |J_1|_{19}$ *and* $Vo(G) = Vo(J_1)$*. Then* $G \cong J_1$*.*

Proof. $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ **and** $Vo(J_1) = \pi^*(J_1) = \{2, 3, 5, 6, 7, 11, 15, 19\}$ **(see the Atlas[9]). Clearly,** $GK(J_1)$ **has four connected components:**$\{2, 3, 5\}$, $\{7\}$, $\{11\}$ **and** $\{19\}$. **By the hypothesis we have** $Vo(G) = Vo(J_1)$. **Then by Theorem 3.1 we conclude that** G **has a normal series** $K < M \leq G$ **such that** K **is the maximal solvable normal subgroup of** G, M/K **is a simple group and** $G/K \leq Aut(M/K)$. **Moreover,** $\pi(G) = \pi(J_1) = \{2, 3, 5, 7, 11, 19\}$, $\Gamma(G) = \Gamma(J_1) = GK(J_1)$ **and** $n(GK(M/K)) \geq 4$.

Clearly, we have that $|\pi(M/K)| \geq 4$. **Suppose that** $|\pi(M/K)| = 4$. **Then** $n(GK(M/K)) = 4$, **and thus the order of every element of** M/K **is a prime power. It follows by [25, Table 3] that** $M/K \cong L_3(4)$. **We have that** $9 \in Vo(L_3(4))$**(see the Atlas[9]). Hence, by Lemma 1.2, Lemma 1.4 and Theorem 3.1(1), we have that** $9 \in \pi^*(J_1) = \{2, 3, 5, 6, 7, 11, 15, 19\}$, **a contradiction. So, we have that** $|\pi(M/K)| \geq 5$. **Then, noting that** $\pi(M/K) \subseteq \{2, 3, 5, 7, 11, 19\}$, **by [22, Table 1] we conclude that** M/K **is isomorphic to one of the following groups:** $M_{22}, A_{11}, McL, Hs, A_{12}, U_6(2), U_3(19), L_4(7), J_1, L_3(11)$ **and** HN. **Then by [25, Table 3] we get that** $M/K \cong M_{22}$ **or** J_1.

Assume that $M/K \cong M_{22}$. **Since** $G/K \leq Aut(M/K) = Aut(M_{22})$, **we have that** $G/K \cong M_{22}$ **or** $M_{22}.2$ **(see the Atlas[9]). If** $G/K \cong M_{22}.2$, **then** $14 \in Vo(G/K)$ **(see the Atlas[9]), and so by Theorem 3.1(1) we have that** $14 \in \pi^*(J_1) = \{2, 3, 5, 6, 7, 11, 15, 19\}$, **a contradiction. Hence, we have that** $G/K \cong M_{22}$. **It follows that** $|G/K| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ **(see the Atlas[9]). Then by the hypothesis we have that** $|K|_{19} = |G|_{19} = |J_1|_{19} = 19$. **So, by using the same argument as in (II) of the proof of Theorem 4.9 we will get a contradiction.**

Assume that $M/K \cong J_1$. **Since** $G/K \leq Aut(M/K) = Aut(J_1)$ **and** $|Out(J_1)| = 1$ **(see the Atlas[9]), we have that** $G/K \cong J_1$. **We know that, for a group** H, **if** $\pi_e(H) = \pi_e(J_1)$, **then** $H \cong J_1$ **(see [23, Theorem 2.7]). Therefore, by Theorem 3.3 we have that** $K = 1$ **and** $G \cong J_1$. **This completes the proof of the theorem.**

Theorem 4.11. *Assume that* $|G|_{\{3,7,11\}} = |U_6(2)|_{\{3,7,11\}}$ *and* $Vo(G) = Vo(U_6(2))$. *Then* $G \cong U_6(2)$.

Proof. Note that $U_6(2) = {}^2A_5(2)$. **We have that** $|U_6(2)| = 2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ **and** $GK(U_6(2))$ **has three connected components:** $\{2, 3, 5\}$, $\{7\}$ **and** $\{11\}$**(see the Atlas[9]). Then, since** $Vo(G) = Vo(U_6(2))$ **by the hypothesis, by Theorem 3.1 we conclude that** G **has a normal series** $K < M \leq G$ **such that** K **is the maximal solvable normal subgroup of** G, M/K **is a simple group and** $G/K \leq Aut(M/K)$. **Moreover,** $\pi(G) = \pi(U_6(2)) = \{2, 3, 5, 7, 11\}$, $\Gamma(G) = \Gamma(U_6(2)) = GK(U_6(2)$ **and** $n(GK(M/K)) \geq 3$. **It follows that** $\pi(M/K) \subseteq \{2, 3, 5, 7, 11\}$, **and** $|\pi(M/K)| = 3, 4$ **or** 5.

(I) Assume that $|\pi(M/K)| = 3$.

Noting that $n(GK(M/K)) \geq 3$, **by Table 1** M/K **is isomorphic to the following groups:**A_5, $L_2(7)$, $L_2(8)$ **and** A_6. **By the hypothesis we have that** $|G|_{\{3,11\}} = |U_6(2)|_{\{3,11\}} = 3^6 \cdot 11$. **So, by using the same argument as in (I) of the proof of Theorem 4.2 we will get a contradiction.**

(II) Assume that $|\pi(M/K)| = 4$.

Noting that $\pi(M/K) \subseteq \{2, 3, 5, 7, 11\}$, **by [22, Table 1] we have that either** $\pi(M/K) = \{2, 3, 5, 7\}$ **or** $\{2, 3, 5, 11\}$.

Assume that $\pi(M/K) = \{2, 3, 5, 7\}$. **Noting that** $n(GK(S_4(7)) = 2$ **(see [25, Table 1]) and** $n(GK(M/K)) \geq 3$, **by [22, Table 1] we conclude that** M/K **is isomorphic to one of the following groups:** $A_7.L_2(49)$, $U_3(5)$, $L_3(4)A_8$, A_9, J_2, A_{10}, $U_4(3)$, $S_6(2)$ **and** $O_8^+(2)$. **Then, since** $G/K \leq Aut(M/K)$. **by checking in the Atlas[9] we conclude that** $|K|_{11} = 11$. **So, by using the same argument as in (I) of the proof of Theorem 4.2, we will get a contradiction.**

Assume that $\pi(M/K) = \{2, 3, 5, 11\}$. **Then, by [22, Table 1] we conclude that** M/K **is isomorphic to one of the following groups:**$L_2(11)$, M_{11}, M_{12} **and** $U_5(2)$. **Hence, noting that** $n(GK(M/K)) \geq 3$, **by [25, Table 2 and Table 3] we conclude that** $M/K \cong L_2(11)$ **or** M_{11}.

Assume that $M/K \cong L_2(11)$. **Since** $G/K \leq Aut(M/K) = Aut(L_2 11)$ **and** $|Out(L_(11)| = 2$ **(see the Atlas[9]), We have that** $|G/K| = 2^2 \cdot 3 \cdot 5 \cdot 11$ **or** $2^3 \cdot 3 \cdot 5 \cdot 11$ **(see the Atlas[9]),**$\pi(K) \subseteq \{2, 3, 5, 7\}$ **and** $|K|_7 = 7$. **So, by using the same argument as in (I) of the proof of Theorem 4.2 we will get a contradiction.**

Assume that $M/K \cong M_(11)$. **Since** $G/K \leq Aut(M/K) = Aut(M_{11})$ **and** $|Out(M_{11})| = 1$ **(see the Atlas[9]), we have that** $G/K = M/K \cong M_{11}$. **It follows that** $|G/K| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, $\pi(K) \subseteq \{2, 3, 5, 7\}$ **and** $|K|_7 = 7$. **So, by using the same argument as in (I) of the proof of Theorem 4.2 we will get a contradiction.**

(III) Assume that $|\pi(M/K)| = 5$.

In this case, $\pi(M/K) = \{2, 3, 5, 7, 11\}$. Noting that $n(GK(M/K)) \geq 3$, by [25, Table 2 and Table 3] we conclude that $M/K \cong M_{22}$ or $U_5(2)$.

Assume that $M/K \cong M_{22}$. Since $G/K \leq Aut(M/K) = Aut(M_{22})$ and 7 is an isolated vertex of $\Gamma(G)(= \Gamma(U_6(2) = GK(U_6(2)))$, by checking in the Atlas[9] we conclude that $G/K = M/K \cong M_{22}$. It follows that $|G/K| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, $\pi(K) \leq \{2, 3, 5\}$ and $|K|_3 = 3^4$. Let P be a Sylow 3-subgroup of K. Then $|Aut(\Omega(Z(P)))| | (3^4 - 1)(3^3 - 1)(3^2 - 1)(3 - 1)$. So, by using the same argument as in (I) of the proof of Theorem 4.x we will get a contradiction.

Assume that $M/K \cong U_6(2)$. Since $G/K \leq Aut(M/K) = Aut(U_6(2))$ and 7 and 11 are isolated vertex of $\Gamma(G)(= GK(U_6(2))$, by checking in the Atlas[9]) we conclude that $G/K \cong M/K \cong U_6(2)$. We know that, for a group H, if $\pi_e(H) = \pi_e(U_6(2)$, then $H \cong U_6(2)$(see [24, Table 1]), and so by Theorem 3.3 we conclude that $K = 1$ and $G \cong U_6(2)$. This completes the proof of the theorem. 2

Theorem 4.12. *Let $S = O_8^-(2)$ or $S_8(2)$. Assume that $|G| = |S|$ and $Vo(G) = Vo(S)$. Then $G \cong S$.*

Proof. We only investigate the case $S = O_8^-(2)$; For the case $S = S_8(2)$, the proof is similar.

We have that $|O_8^-(2)| = 2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$, $Vo(O_8^-(2)) = \pi_e^*(O_8^-(2)) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 17, 21, 30\}$ (see the Atlas[9]). Clearly, $GK(O_8^-(2))$ has two connected components: $\pi_1 = \{2, 3, 5, 7\}$ and $\pi_2 = \{17\}$, and π_1 is not a complete graph. By the hypothesis we have that $Vo(G) = Vo(O_8^-(2))$. Then by Theorem 3.2 we conclude that G has a normal series $K < M \leq G$ such that K is the maximal solvable normal subgroup of G, M/K is a simple group and $G/K \leq Aut(M/K)$.

By the hypothesis we have that $|G| = |O_8^-(2)| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$. It follows that $\pi(M/K) \subseteq \{2, 3, 5, 7, 17\}$ and $|\pi(M/K)| = 3, 4$ or 5.

(I) Assume that $|\pi(M/K)| = 3$.

By Table 1 we conclude that M/K is isomorphic to one of the following groups: A_5, $L_2(7)$, $L_2(8)$, $L_2(17)$, $U_3(3)$, $U_4(2)$ and A_6. Then by using the same argument as in (I) of the proof of Theorem 4.2 we will get a contradiction.

(II) Assume that $|\pi(M/K)| = 4$.

by [22, Table 1] M/K is isomorphic to one of the following groups: A_7, $L_3(4)$, A_8, A_9, $U_4(3)$, $S_6(2)$ and $L_2(16)$. Then by using the same argument as in (II) of the proof of Theorem 4.9 we will get a contradiction.

(III) Assume that $|\pi(M/K)| = 5$.

In this case $\pi(M/K) = \{2, 3, 5, 7, 17\}$. Noting that $|G| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$, by [22, Table 1] $M/K \cong O_8^-(2)$. Then, since $|G| = |O_8^-(2)|$, we get that $G \cong O_8^-(2)$. This completes the proof of the theorem. 2

We have that $|L_5(2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$, and $Vo(L_5(2)) = \pi_e^*(L_5(2)) = \{2, 3, 4, 5, 6, 7, 8, 12, 14, 15, 21, 31\}$ (see the Atlas[9]). Clearly, $GK(L_5(2))$ has two connected composition: $\pi_1 = \{2, 3, 5, 7\}$ and $\pi_2 = \{31\}$, and π_1 is not a complete graph. So, by using the same argument as in the proof of Theorem 4.12, we conclude that the following theorem holds:

Theorem 4.13. *Assume that $|G|_{\{5,7,31\}} = |L_5(2)|_{\{5,7,31\}}$ and $Vo(G) = Vo(L_5(2))$. Then $G \cong L_5(2)$.*

5 Some results on Problem B

In this section, we establish several results on Problem B, that is, on V-recognition of a simple group. We already know that the simple groups $L_2(2^a)$ and $Sz(2^{2n+1})$ are V-recognizable (see Section 1).

Theorem 5.1. *Assume that $Vo(G) = Vo(L_2(23))$. Then $G \cong L_2(23)$, that is, $L_2(23)$ is V-recognizable.*

Proof. We have that $|L_2(23)| = 2^3{\cdot}3{\cdot}11{\cdot}23$ and $\pi_e^*(L_2(23)) = Vo(L_2(23)) = \{2, 3, 4, 6, 11, 12, 23\}$ (see the Atlas[9]). Clearly, $n(GK(L_2(23))) = 3$. Then, since $Vo(G) = Vo(L_2(23))$ by the hypothesis, by Theorem 3.1 G has a normal series $K < M \leq G$ such that K is the maximal

solvable normal subgroup of G, M/K is a simple group and $G/K \leq Aut(M/K)$. Moreover, $\pi(G) = \pi(L_2(23)) = \{2, 3, 11, 23\}$, $\Gamma(G) = \Gamma(L_2(23)) = GK(L_2(23))$ and $n(GK(M/K)) \geq 3$. It follows that $\pi(M/K) \subseteq \{2, 3, 11, 23\}$. Clearly, either $|\pi(M/K)| = 3$ or $\pi(M/K) = \{2, 3, 11, 23\}$. By Table 1 we conclude that $|\pi(M/K)| \neq 3$, and so $\pi(M/K) = \{2, 3, 11, 23\}$. It follows by Table 1 in [22] that $M/K \cong L_2(23)$. Then, since $G/K \leq Aut(M/K) = Aut(L_2(23))$, by checking in the Atlas[9] we conclude that $G/K = M/K \cong L_2(23)$. We know that, for a group H, if $\pi_e(H) = \pi_e(L_2(23))$, then $H \cong L_2(23)$(see [23, Theorem 2.7]). So, by Theorem 3.3 we conclude that $K = 1$ and $G \cong L_2(23)$. The proof is finished. 2.

By Table 1 in [22] and by checking in the Atlas [9], we obtain the following Table 4.

Table 4 Simple groups G with $\pi(G) = \{2, 3, 7, 13\}$.

G	$\|G\|$	$Vo(G) = \pi_e^*(G)$	$V(\Gamma(G))$	$n(\Gamma(G))$
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	{2, 3, 7, 13, 14}	{2, 3, 7, 13}	3
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	{2, 3, 4, 6, 7, 8, 9, 12, 13}	{2, 3, 7, 13}	3
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	{2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 18, 21, 28}	{2, 3, 7, 13}	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	{2, 3, 6, 7, 13}	{2, 3, 7, 13}	3

Theorem 5.2. *Assume that $Vo(G) = Vo(L_2(27))$. Then $G \cong L_2(27)$, that is, $L_2(27)$ is V-recognizable.*

Proof. By Table 4, we have that $\pi^*_e(L_2(27)) = Vo(L_2(27)) = \{2, 3, 7, 13, 14\}$. Clearly, $GK(L_2(27))$ has three connected components: $\{2, 7\}$, $\{3\}$ and $\{13\}$. By the hypothesis we have that $Vo(G) = Vo(L_2(27))$. It follows by Theorem 3.1 that G has a normal series $K < M \leq G$ such that K is the maximal solvable normal subgroup of G, M/K is a simple group, $G/K \leq Aut(M/K)$. Moreover, $\pi(G) = \pi(L_2(27)) = \{2, 3, 7, 13\}$, $\Gamma(G) = \Gamma(L_2(27)$ and $n(GK(M/K)) \geq 3$. In addition, if M/K is a simple group of Lie type, then $\pi_e(M/K) \subseteq \pi_e(L_2(27)) = \{1, 2, 3, 7, 13, 14\}$. Then, by Table 1 and Table 4 we conclude that $M/K \cong L_2(27)$. Then, since $G/K \leq Aut(M/K) = Aut(L_2(27))$ and each element in $Vo(G/K)$ is a factor of some element in $Vo(G) = Vo(L_2(27)) = \{2, 3, 7, 13, 14\}$(see Lemma 1.8), by checking in the Atlas [9] we conclude that $G/K = M/K \cong L_2(27)$. We know that, for a group H, if $\pi_e(H) = \pi_e L_2(27)$, then $H \cong L_2(27)$(see [23, Theorem 2.7]). So, by Theorem 3.3 we conclude that $K = 1$ and $G \cong L_2(27)$. This completes the proof of the theorem. 2

By using the same argument as in the proof of Theorem 5.2, we conclude that the following theorem holds:

Theorem 5.3 *Assume that $Vo(G) = Vo(L_2(13))$. Then $G \cong L_2(13)$, that is, $L_2(13)$ is V-recognizable.*

Theorem 5.4 *Assume that $Vo(G) = Vo(J_4)$. Then $G \cong J_4$, that is, J_4 is V-recognizable.*

Proof. Let S be a simple group. Then $n(GK(S)) \leq 6$ (see[8]). We have that $n(GK(J_4)) = 6$ (see [25, Table 3]). Then, since $Vo(G) = Vo(J_4)$ by the hypothesis, by Theorem 3.1 G has a normal series $K < M \leq G$ such that K is the maximal solvable normal subgroup of G, M/K is a simple and $G/K \leq Aut(M/K)$. Moreover, $\pi(G) = \pi(J_4)$,$\Gamma(G) = \Gamma(J_4)) = GK(J_4))$ and $n(GK(M/K)) = 6$. Then, since $n(GK(M/K)) = 6$, we have that $M/K \cong J_4$ (see [25, Table 3]). Hence, since $G/K \leq Aut(M/K) = Aut(J_4)$ and $|Out(J_4)| = 1$ (see the Atlas[9]), we get that $G/K \cong J_4$. We know that, for a group H, if $\pi_e(H) = \pi_e(J_4)$, then $H \cong J_4$ (see [23,Theorem 2.7]).So, by Theorem 3.3 we have that $K = 1$ and $G \cong J_4$. The proof is finished. 2.

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Referenses

[1] B. Huppert, Character Theory of Finite Groups, Berlin: Walter der Gruyter, 1998.

[2] S. Dolfi, E. Pacific, L. Sanus and P. Spiga, On the vanishing prime graph of solvable groups, J. Group Theory 13(2010),189-206.

[3] Foroudi Ghasemabadi M., Iranmanesh A., Ahanjideh M., A new characterization of some families of finite simple groups, Rend.Semin. Mat. Univ. Padova, 2017,137:57-74.

[4] F. Ghasemabadi, A. Iranmanesh, F. Mavadapour, A new characterization of some finite simple groups, Sibirsk. Math. Zh. 56(2015),no.1,94-99.

[5] S. Dolfi, F. pacifici, L. Samts and P.Spiga, On the orders of zeros of irreducible characters, J. Algebra, 321(2009),345-352.

[6] I. M. Isaacs, Character Theory of Finite groups, New York, 1976.

[7] A. Granville and K. Ono, Defect zero p-blocks for finite simple groups, Trans. Amer. Math. Soc., 348(1): 331-347, 1996.

[8] S. Dolfi, E. Pacifici, L. Sanus and P. Spiga, On the vanishing prime graph of finite groups, J. London Math.Soc.,(2)82(2010), 167-183.

[9] J.H.Conway, R.T.Curtis, S.P. Norton, et al. Atlas of finite groups, Oxford: Clarendon Press, 1985.

[10] M. Suzuki, finite groups with nilpotent centralizers, Trans. Amer. Math. Soc., 99(1961) 125-170.

[11] P. J. Cameron and N. V. Maslova, Criterion of unrecognizability of a finite group by its Gruenberg-Kegel graph, J. algebra, 607(2022), Part A,186-213.

[12] L. Dornhoff, Group Representation Theory, PART A, MARCEL DEKKER, INC. 1971.

[13] B. Huppert, Endlish Gruppen I, Springer-Verlag, Berlin Heidelberg 1967.

[14] W. J. Shi, A characteristic property of J_1 and $L_2(2^a)$, Adv.Math.(China), 16(1987),397-401.

[15] B. Huppert and N. Blackburn, Finite Groups III, Springer-Verlag, 1982.

[16] W. J.Shi, A characterization of Suzuki's simple groups, Proc. Amer. Math. Soc., 3(1992),589-591.

[17] Zhang J. S, Recognition of $PSL(2^a)$ by the orders of vanishing elements, Proc. Indian. Acad. Sci. Math. Sci., 2018, 128(5):Paper No. 58,6pp.

[18] Zhang J. S.,Shao Z.C., Shen Z. C., A new characterzation of of Suzuki's simple groups, J. Algebra Appl., 2017, 16(11): 1750216, 6pp.

[19] Liu D. D., Zhang J.S., Recognition of some finite simple groups by the orders of vanishing elements, Proc. Indian Acad. Sci. Math. Sci., 2021,1311Paper No.5.5,9 pp.

[20] V. D. Mazunov, Groups containing a self-centralizing subgroup of order 3, Algebra and Logic, 42(2003),29-36.

[21] D.Gorenstein, Finite Simple Groups, Harper Row, 1968.

[22] A. V. Zavarnitsine, Finite simple groups with narrow primes spectrum, Siberian Electronic mathematical Reports, 6(2009),1-12.

[23] Wujie Shi, Quantitative characterization of finite groups:A complement, Int. J. Group Theorem, 14, No. 4 (2025), 181-222.

[24] A.V.Vasil'ev, On recognition of all finite nonabelian simple groups with orders having prime divisors at most 13, Siberian Mathematical Journal, 46, No.2 (2005), 246-253.

[25] Y. Chen and G. Chen, Recognizing $L_2(p)$ by its order and one special conjugacy class ziz, Journal of Inequalities and Applications,310(2012), 2-10 .